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Quasi-functors as lifts of Fourier-Mukai functors: the uniqueness problem

Relatore

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Chapter zero

Dear reader,
I'm glad to introduce you to my doctoral thesis. It is an account of about three years of my studying and learning how to do research in mathematics, working on one of its most beautiful topics, *category theory* – with an eye to applications to geometry. It would be great to write an entire book as an introduction to the contents of this work, explaining category theory from its very foundations, but this is not permitted in a Ph.D. thesis, which has instead to be a rather technical exposition of the results obtained. So, I must assume that you are already well acquainted with categories and functors, and also some more advanced topics, which I will mention in the sections of this "zeroth chapter".

The purpose of this introduction is twofold: on one hand, it will contain a summary of the main theorems I have managed to prove, together with their theoretical background and their motivation; on the other hand, it will be the occasion to write down some definitions and results which we shall refer to in the following chapters. This explains why I have decided to label it as "Chapter zero", as if it were actually part of the entire work. Starting from the next section, I will adopt the more usual detached style that you probably are familiar with, if you are used to reading mathematical works. Nonetheless, the style will sometimes be mathematically "informal": you will encounter "pre-definitions" and "pre-theorems", which are just informal statements written to give you some intuition; in any case don't worry, since I will always refer to the precise definitions and theorems that you will find in the thesis. Have a good reading!

0.1 Dg-categories and quasi-functors

Triangulated categories are nowadays a classical topic in mathematics, with many applications in geometry and algebra. We shall assume that the reader is acquainted with the definitions and the basic features of the theory, and the classical examples (derived categories of abelian categories). The well-known drawback of triangulated categories is the *non-functoriality of cones* of morphisms. Cones should themselves be homotopy colimits, and satisfy a suitable universal property; this suggests that a triangulated category is, in some sense, a "shadow" of a more complicated, *higher categorical* structure, where the notion of *homotopy* is formalised and becomes functorial. Among the many models of higher categories, one of the most suitable to our needs (and the main object of study of this thesis) is given by *differential graded (dg-) categories.*

Pre-definition 0.1.1 (Definitions 1.1.1, 1.1.9 and 1.2.1)**.** A dg-category $\mathbf{A}$ is a category whose hom-sets are endowed with a structure of cochain complex over a ground commutative ring $\mathbf{k}$, such that the compositions are (associative, unital) chain maps. A dg-functor $F\colon \mathbf{A} \to \mathbf{B}$ between dg-categories is a functor which preserves the additional structure, namely, the map on morphisms

$$F\colon \mathbf{A}(A, B) \to \mathbf{B}(F(A), F(B))$$

is a chain map. Given two dg-categories $\mathbf{A}$ and $\mathbf{B}$, the category $\operatorname{Fun}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})$ of dg-functors has a natural structure of dg-category.

Dg-categories are based on complexes of $\mathbf{k}$-modules. This choice puts them in the realm of homological algebra, where we have many important homotopical features, and the advantage of a relevant "computational simplicity" over other models (such as topological ones). The category of complexes can be viewed itself as a dg-category, called $\mathsf{C}_{\mathrm{dg}}(\mathbf{k})$. The homotopical features of complexes (chain homotopies above all) and other relevant constructions can be translated in the theory of dg-categories. A fundamental one is clearly given by taking cohomology. Namely, given a dg-category $\mathbf{A}$, we may form the graded category of graded cohomology $H^*(\mathbf{A})$ and the ordinary $\mathbf{k}$-linear category $H^0(\mathbf{A})$ of zeroth cohomology: take the same objects as $\mathbf{A}$ and just project the compositions. The construction is functorial, and also gives $H^*(F)$ and $H^0(F)$ for a given dg-functor F. It should be pointed out that the dg-category of complexes $\mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ is defined in such a way that two complexes are homotopy equivalent if and only if they are isomorphic in $H^0(\mathsf{C}_{\mathrm{dg}}(\mathbf{k}))$. So, we have a consistent notion of *homotopy equivalence* in any dg-category: A and B are homotopy equivalent if they are isomorphic in $H^0(\mathbf{A})$.

Other important constructions that can be performed on chain complexes are taking *shifts* and *mapping cones*. The reader can expect that both can be translated to any dg-category: see Definitions 2.3.1 and 2.3.2. Both shifts and cones are characterised by means of universal properties, that is, they are *functorial* (actually, they are particular homotopy colimits). Now, it should be quite clear how dg-categories can be employed to enhance triangulated categories:

Pre-definition 0.1.2 (Definition 2.3.13)**.** Let $\mathbf{A}$ be a dg-category. We say that $\mathbf{A}$ is *pretriangulated* if it contains, *up to homotopy equivalence*, all shifts of objects $A[n]$ and all functorial cones $\mathrm{C}(f)$ of closed degree 0 morphisms.

In the philosophy of homotopy theory and higher category theory, the existence of shifts and (functorial) cones in $\mathbf{A}$ is required to hold up to homotopy equivalence. The following fundamental result is now a matter of rather straightforward verifications:

Pre-theorem 0.1.3 (Theorem 2.3.14)**.** *If* $\mathbf{A}$ *is a pretriangulated dg-category, then* $H^0(\mathbf{A})$ *has a natural structure of triangulated category.*

Given a triangulated category $\mathbf{T}$, it is important to decide whether there exists a pretriangulated dg-category $\mathbf{A}$ such that $H^0(\mathbf{A}) \cong \mathbf{T}$: such a dg-category will be called an *enhancement* of $\mathbf{T}$. In order to address the (likewise important) problem of *uniqueness* of enhancements, we first have to understand what is the "right way" (from the homotopical point of view) to identify two given dg-categories. The correct notion is that of *quasi-equivalence*:

Pre-definition 0.1.4 (Definition 1.3.2)**.** A dg-functor $F \colon \mathbf{A} \to \mathbf{B}$ is a *quasi-equivalence* if F induces quasi-isomorphisms between the corresponding hom-complexes, and $H^0(F)$ is an equivalence.

In the case when our dg-categories are pretriangulated, a quasi-equivalence F is just a dg-functor such that $H^0(F)$ is an equivalence (Lemma 2.3.16). Quasi-equivalences clearly induce a relation in the collection of dg-categories, but there is a serious technical drawback: this relation is not symmetric. In other words, if there is a quasi-equivalence $\mathbf{A} \to \mathbf{B}$, there doesn't need to be a quasi-equivalence $\mathbf{B} \to \mathbf{A}$. Dg-functors themselves are not "homotopy meaningful": they are defined just as ordinary functors, whereas a "homotopy coherent" dg-functor should *not* satisfy a strict identity $F(gf) = F(g)F(f)$, but should instead satisfy something like "$F(gf)$ is coherently homotopic to $F(g)F(f)$". So, the idea is that any inverse of a quasi-equivalence should be such "homotopy coherent dg-functor". Extensive work has been done (by B. Toën and G. Tabuada among others) to make this precise. The result is the so-called *homotopy theory of dg-categories*:

Pre-theorem 0.1.5 ([Tab05], [Toë07], Theorems 1.3.5 and 1.3.10)**.** *The category* dgCat *of small dg-categories has a model category structure whose weak equivalences are the quasi-equivalences. Moreover, its localisation along quasi-equivalences, called* Hqe*, admits an internal hom: for all dg-categories* $\mathbf{A}$ *and* $\mathbf{B}$*, there is a dg-category* $\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B})$*, defined up to quasi-equivalence, which enhances the hom-set* $\mathsf{Hqe}(\mathbf{A}, \mathbf{B})$ *in the following sense:*

$$\mathsf{Hqe}(\mathbf{A}, \mathbf{B}) \leftrightarrow \{\textit{isom. classes of objects of } H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B}))\}.$$

As a matter of terminology, we call *quasi-functor* an object of $\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B})$. Objects in $H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B}))$ which admit an inverse in $H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{B}, \mathbf{A}))$ correspond essentially to quasi-equivalences, so the above theorem is a satisfactory result, at least from the theoretical point of view: the relevant morphisms between dg-categories are precisely the quasi-functors. Now, the difficulty is to *work concretely* with them. Being higher categorical entities, they are intrinsically complicated. However, there are some "concrete incarnations" of the dg-category $\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B})$ which enable us to do computations, at least to some extent. If $\mathbf{k}$ is a field, $\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B})$ can be described concretely by something that actually formalises precisely the idea of "homotopy coherent dg-functor": namely, A_∞-functors (see Chapter 5). Their drawback lies in the complexity of the formulae involved. Another way to describe $\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B})$ is by employing *bimodules*, as we are going to explain.

Quasi-functors as bimodules; adjoints

Given a dg-category $\mathbf{A}$, for simplicity we set

$$\mathsf{C}_{\mathrm{dg}}(\mathbf{A}) = \mathrm{Fun}_{\mathrm{dg}}(\mathbf{A}^{\mathrm{op}}, \mathsf{C}_{\mathrm{dg}}(\mathbf{k})).$$

The dg-category $\mathsf{C}_{\mathrm{dg}}(\mathbf{A})$ is the target of the Yoneda embedding of $\mathbf{A}$. If $F\colon \mathbf{A} \to \mathbf{B}$ is a dg-functor, then composing it with the Yoneda embedding $\mathbf{B} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{B})$, we obtain a dg-functor

$$h_F\colon \mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{B}). \tag{0.1.1}$$

This is a very special example of $\mathbf{A}$-$\mathbf{B}$-*dg-bimodule* (see Definition 1.2.5), with the property that it actually comes from a genuine dg-functor $\mathbf{A} \to \mathbf{B}$: namely, $h_F(A)$ is *representable* for any $A \in \mathbf{A}$. So, in order to define quasi-functors, we may try to follow this pattern, employing some kind of "weak representability" notion. Fortunately, we have a natural (componentwise) notion of *quasi-isomorphism* in the dg-category $\mathsf{C}_{\mathrm{dg}}(\mathbf{B})$; hence, we give the following definition:

Pre-definition 0.1.6 (Section 3.5)**.** A *quasi-functor* $F\colon \mathbf{A} \to \mathbf{B}$ can be defined as a bimodule $F\colon \mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{B})$ such that $F(A)$ is quasi-isomorphic to a representable dg-functor $\mathbf{B}^{\mathrm{op}} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ for all $A \in \mathbf{A}$. We also say that F is *right quasi-representable.*

Up to some technical issues, the above notion defines the objects of $\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B})$. Now, an obvious problem is recollecting the features of ordinary category theory, in this particular context of dg-categories and quasi-functors. What we have managed to do in this thesis is to give a simple characterisation of *adjunctions* of quasi-functors (themselves, they are defined as adjunctions in a suitable bicategory of bimodules, see the subsection 'Adjoints' of Section 3.5). To have a grasp of the idea, start from an ordinary adjunction of dg-functors $F \dashv G\colon \mathbf{A} \to \mathbf{B}$: it is an isomorphism

$$\mathbf{B}(F(A), B) \cong \mathbf{A}(A, G(B)),$$

natural in $A \in \mathbf{A}$ and $B \in \mathbf{B}$. This naturality implies that this is actually an isomorphism of bimodules: on the right hand side we have the $\mathbf{B}$-$\mathbf{A}$-bimodule h_G, whereas on the left hand side we have the bimodule h^F, which is obtained from $h_F\colon \mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{B})$ by means of a sort of *duality*:

$$h_F(A)(B) = \mathbf{B}(B, F(A)),$$
$$h^F(B)(A) = \mathbf{B}(F(A), B)).$$

This duality is actually defined for all bimodules: it maps functorially $\mathbf{A}$-$\mathbf{B}$-bimodules to $\mathbf{B}$-$\mathbf{A}$-bimodules, and vice-versa (see Proposition 3.3.4). Bimodules of the form h_F (up to isomorphism of bimodules), which are called *right representable*, are mapped to bimodules of the form h^F (up to isomorphism), which are called *left representable.* Clearly, saying "the dg-functor G has a left adjoint" is equivalent to saying "h_G is left representable": $h^F \cong h_G$. Upon overcoming some technical difficulties (addressed by means of the duality construction we have mentioned) it can be proved that this characterisation can be consistently extended to quasi-functors in the natural way. First, one has to define *left quasi-representable* bimodules in the obvious way; then, we have:

Pre-theorem 0.1.7 (Proposition 3.5.5)**.** *Let $G\colon \mathbf{B} \to \mathbf{A}$ be a quasi-functor. Then, G has a left adjoint quasi-functor if and only if it is left quasi-representable.*

As the reader may expect, there is a similar characterisation of right adjoints. The result can be applied to prove an existence theorem of adjoint quasi-functors, under some hypotheses on the dg-categories:

Pre-theorem 0.1.8 (Theorem 3.5.9)**.** *Let $\mathbf{A}, \mathbf{B}$ be dg-categories. Assume that $\mathbf{A}$ is triangulated and smooth, and that $\mathbf{B}$ is locally perfect. Let $T \colon \mathbf{A} \to \mathbf{B}$ be a quasi-functor. Then, T admits both a left and a right adjoint.*

0.2 Dg-lifts and Fourier-Mukai kernels

Quasi-functors, as the reader may expect, yield ordinary functors by taking cohomology. Namely, there is a functor:

$$H^0 = \Phi^{\mathbf{A} \to \mathbf{B}} \colon H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B})) \to \mathrm{Fun}(H^0(\mathbf{A}), H^0(\mathbf{B})). \tag{0.2.1}$$

If $\mathbf{A}$ and $\mathbf{B}$ are pretriangulated, then $\Phi^{\mathbf{A} \to \mathbf{B}}$ is viewed as taking values in the category of exact functors $\mathrm{Fun}_{\mathrm{ex}}(H^0(\mathbf{A}), H^0(\mathbf{B}))$. By definition, a *dg-lift* of a functor $\overline{F} \colon H^0(\mathbf{A}) \to H^0(\mathbf{B})$ is a quasi-functor $F \colon \mathbf{A} \to \mathbf{B}$ such that $H^0(F) = \overline{F}$. The *uniqueness problem of dg-lifts*, which is the main topic of the thesis, amounts to studying whether $\Phi^{\mathbf{A} \to \mathbf{B}}$ is essentially injective. The relevance of this problem lies in the fact that it is essentially equivalent to the *uniqueness problem of Fourier-Mukai kernels*, which is of current interest in algebraic geometry. Let us make this claim precise. From now on assume that the ground commutative ring $\mathbf{k}$ is a field.

Let X be a quasi-compact and quasi-separated scheme (over $\mathbf{k}$). We denote by $\mathfrak{D}(\mathrm{QCoh}(X))$ the derived category of quasi-coherent sheaves on X. The subcategory of compact objects of $\mathfrak{D}(\mathrm{QCoh}(X))$ coincides with the category of perfect complexes $\mathrm{Perf}(X)$. Given two schemes X and Y, there is a functor:

$$\Phi_{-}^{X \to Y} \colon \mathfrak{D}(\mathrm{QCoh}(X \times Y)) \to \mathrm{Fun}_{\mathrm{ex}}(\mathrm{Perf}(X), \mathfrak{D}(\mathrm{QCoh}(Y))), \tag{0.2.2}$$

which maps a complex $\mathcal{E} \in \mathfrak{D}(\mathrm{QCoh}(X \times Y))$ to its *Fourier-Mukai functor*

$$\Phi_{\mathcal{E}} \colon \mathrm{Perf}(X) \to \mathfrak{D}(\mathrm{QCoh}(Y)).$$

If an exact functor $F \colon \mathrm{Perf}(X) \to \mathfrak{D}(\mathrm{QCoh}(Y))$ is such that $F \cong \Phi_{\mathcal{E}}$, we say thay $\mathcal{E}$ is a *Fourier-Mukai kernel* of F. Current research is devoted to investigating the properties of $\Phi_{-}^{X \to Y}$ (see [CS12a] for a survey); for instance, the uniqueness problem of Fourier-Mukai kernels is equivalent to the essential injectivity of $\Phi_{-}^{X \to Y}$.

Now, let us see how this is related to dg-categories and quasi-functors. If X is a quasi-compact and quasi-separated scheme (over the field $\mathbf{k}$), then the derived category $\mathfrak{D}(\mathrm{QCoh}(X))$ has an enhancement, which we call $\mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(X))$, choosing it once and for all and identifying $H^0(\mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(X))) = \mathfrak{D}(\mathrm{QCoh}(X))$. Taking the dg-subcategory of $\mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(X))$ whose objects correspond to $\mathrm{Perf}(X)$, we find an enhancement $\mathrm{Perf}_{\mathrm{dg}}(X)$ of the category of perfect complexes. A remarkable theorem by B. Toën tells us that, under suitable hypotheses, *every quasi-functor has a unique Fourier-Mukai kernel*, in the following sense:

Theorem 0.2.1 (Adapted from [Toë07, Theorem 8.9])**.** *Let X and Y be quasi-compact and separated schemes over* $\mathbf{k}$*. Then, there is an isomorphism in* Hqe*:*

$$\mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(X\times Y)) \xrightarrow{\sim} \mathbb{R}\underline{\mathrm{Hom}}(\mathrm{Perf}_{\mathrm{dg}}(X), \mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(Y)). \tag{0.2.3}$$

Next, a result adapted from [LS14, Theorem 1.1] gives the desired "bridge" between Fourier-Mukai functors and quasi-functors between dg-categories:

Theorem 0.2.2. *Let X and Y be Noetherian separated schemes over* $\mathbf{k}$ *such that $X\times Y$ is Noetherian and the following condition holds for both X and Y: any perfect complex is isomorphic to a strictly perfect complex (i. e. a bounded complex of vector bundles). Then, there is a commutative diagram (up to isomorphism):*

$$\begin{array}{ccc} \mathfrak{D}(\mathrm{QCoh}(X\times Y)) & \xrightarrow{\sim} & H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathrm{Perf}_{\mathrm{dg}}(X), \mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(Y))) \\ & \searrow^{\Phi_{-}^{X\to Y}} & \downarrow{\scriptstyle \Phi^{\mathrm{Perf}_{\mathrm{dg}}(X)\to \mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(Y))}} \\ & & \mathrm{Fun}_{\mathrm{ex}}(\mathrm{Perf}(X), \mathfrak{D}(\mathrm{QCoh}\,Y)), \end{array} \tag{0.2.4}$$

where the horizontal equivalence is induced by (0.2.3)*.*

Remark 0.2.3. The hypotheses of the above theorem are satisfied if both X and Y are quasi-projective.

The above result tells us that, under suitable hypotheses, the properties of $\Phi_{-}^{X\to Y}$ are directly translated to those of $\Phi^{\mathrm{Perf}_{\mathrm{dg}}(X)\to \mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(Y))}$. In particular, the dg-lift uniqueness problem for functors $\mathrm{Perf}(X) \to \mathfrak{D}(\mathrm{QCoh}(Y))$ (with the above chosen dg-enhancements) is equivalent to the uniqueness problem of Fourier-Mukai kernels.

Next, we show our results, and the corresponding geometric applications. The dg-categories of interest have the feature of being *triangulated* (with this, we mean something more than pretriangulated, see Definition 3.2.4), and being *generated* by some simpler dg-subcategory, which in all cases of our interest will be equivalent to an ordinary $\mathbf{k}$-linear category (see Definition 3.2.7 for the general notion). A first easy result is as follows:

Pre-theorem 0.2.4 (Theorem 4.3.8)**.** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be triangulated dg-categories, and assume that* $\mathbf{A}$ *is generated by a free* $\mathbf{k}$*-linear category* $\mathbf{k}[Q]$ *over a quiver Q. Then,* $\Phi^{\mathbf{A}\to\mathbf{B}}$ *is essentially injective.*

The above result can be proved restricting to the free category $\mathbf{k}[Q]$ of generators, namely, showing that $\Phi^{\mathbf{k}[Q]\to\mathbf{B}}$ is essentially injective. This is done by using the characterisation that quasi-functors $F, G\colon \mathbf{k}[Q] \to \mathbf{B}$ are isomorphic if and only if they are *(right) homotopic* (see Definition 1.3.12): this makes the actual argument of proof very simple. As an application, we notice that the category $\mathrm{Perf}(\mathbb{P}^1)$ has a full and strong two-term exceptional sequence $(\mathcal{O}, \mathcal{O}(1))$, which can be viewed – translated to the corresponding dg-subcategory of $\mathrm{Perf}_{\mathrm{dg}}(\mathbb{P}^1)$ – as a free subcategory of generators. Hence, we have the following result, which was already proved by A. Canonaco and P. Stellari with geometric techniques:

Pre-corollary 0.2.5 (Corollary 4.3.11). *Let Y be a scheme with suitable hypotheses, so that both Theorems 0.2.2 and 0.2.1 are applicable with $X = \mathbb{P}^1$. Then, the functor $\Phi_{-}^{\mathbb{P}^1 \to Y}$ is essentially injective.*

Starting from this, we tried to generalise to dg-categories which have a strong and full exceptional sequence of arbitrary length: a solution of the dg-lift uniqueness problem in this situation would yield a uniqueness result of Fourier-Mukai kernels for functors defined on $\mathrm{Perf}(\mathbb{P}^n)$, which admits the strong and full exceptional sequence $(\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n))$. In general, these exceptional sequences are *nonfree* subcategories of generators; we managed to obtain some positive results, employing a technique based on the notion of *glueing* of dg-categories along a bimodule (see 'The glueing technique' in Section 4.3). Unfortunately, we didn't obtain anything really satisfactory; also, the research of counterexamples has proven itself to be very difficult, and we didn't manage to obtain anything deep (see Section 4.4 for an account of our attempts). Nevertheless, such counterexamples to dg-lift uniqueness exist in general, since there are examples of non-uniqueness of Fourier-Mukai kernels: in particular, when X is an elliptic curve, $\Phi_{-}^{X \to X}$ is not essentially injective, as proved in [CS12b].

A more general dg-lift uniqueness result needs additional assumptions of the functors. What we are able to prove is the following:

Pre-theorem 0.2.6 (Theorem 5.2.4). *Assume that $\mathbf{k}$ is a field. Let $\mathbf{A}$ and $\mathbf{B}$ be triangulated dg-categories. Assume that $\mathbf{A}$ is generated by an ordinary $\mathbf{k}$-linear subcategory $\mathbb{E}$. Moreover, let $F, G \colon \mathbf{A} \to \mathbf{B}$ be quasi-functors satisfying the following vanishing condition:*

$$H^j(\mathbf{B}(F(E), F(E'))) \cong 0,$$

for all $j < 0$, for all $E, E' \in \mathbb{E}$. Then, $H^0(F) \cong H^0(G)$ implies $F \cong G$.

The proof of the above result employs the description of quasi-functors by means of A_∞-functors; even if it involves some rather intricate computations with the A_∞ formalism, it is not conceptually difficult. The geometric application goes as follows:

Pre-theorem 0.2.7 (Theorem 5.3.7). *Let X and Y be schemes satisfying the hypotheses of both Theorems 0.2.2 and 0.2.1, with X quasi-projective. Let $\mathcal{E}, \mathcal{E}' \in \mathfrak{D}(\mathrm{QCoh}(X \times Y))$ be such that*

$$\Phi_{\mathcal{E}}^{X \to Y} \cong \Phi_{\mathcal{E}'}^{X \to Y} \cong F \colon \mathrm{Perf}(X) \to \mathfrak{D}(\mathrm{QCoh}(Y)),$$

and $\mathrm{Hom}(F(\mathcal{O}_X(n)), F(\mathcal{O}_X(m))[j]) = 0$ for all $j < 0$, for all $n, m \in \mathbb{Z}$. Then $\mathcal{E} \cong \mathcal{E}'$.

The above result is an improvement of [CS07, Theorem 1.1], clearly only regarding the uniqueness problem and the non-twisted case: our result holds not only for smooth projective varieties, and with the weaker vanishing hypothesis

$$\mathrm{Hom}(F(\mathcal{O}_X(n)), F(\mathcal{O}_X(m))[j]) = 0.$$

It is also an improvement of [CS14, Remark 5.7], which holds for fully faithful functors.

If we don't put hypotheses on the functors, or if we try to work with dg-categories with more complicated generators, the dg-lift uniqueness problem becomes very complicated, and in general it remains widely unsolved.

0.3 Plan of the work

The thesis is organised in two parts: the first one explains the theoretical bases, whereas the second one deals more specifically with the dg-lift uniqueness problem.

The first two chapters contain the basic foundational material on dg-categories. None of this is original, but the exposition is perhaps somewhat unconventional: in particular, in Chapter 2 we extensively develop *(co)end calculus*, a very useful tool which we hope will become more widespread even among who doesn't study category theory in itself. Chapter 3 deals with the crucial notion of quasi-functor; in particular, we address the notion of *adjoint quasi-functors* and prove a simple characterisation.

Chapter 4 contains the attempts to solve the problem of dg-lift uniqueness in some simple situations, namely, when the domain dg-category is generated by an exceptional sequence. The solution in the free case is contained here, and so are the other attempts, which are based on the 'glueing technique'. Chapter 5, finally, contains the main uniqueness result, which is addressed with A_∞-functors.

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Part I

The theory of dg-categories and quasi-functors

Chapter 1

Basic dg-category theory, I

In this first chapter, we develop the main basic elements of the theory of dg-categories. We will cover the elementary notions and some of the main results on the homotopy theory. We will avoid set theoretic difficulties by implicitly fixing suitable Grothendieck universes. We shall also fix, once and for all, a ground commutative ring $\mathbf{k}$. Virtually every category we shall encounter will be at least $\mathbf{k}$-linear, so we allow ourself some sloppiness, and often employ the terms "category" and "functor" meaning "$\mathbf{k}$-category" and "$\mathbf{k}$-functor".

1.1 Dg-categories and dg-functors

We assume that the reader is acquainted with the theory of (co)chain complexes: we just recollect here some features. We denote by $\mathsf{C}(\mathbf{k})$ the category of (cochain) complexes of $\mathbf{k}$-modules and chain maps. This is a symmetric monoidal category, with unit given by the base ring $\mathbf{k}$ (viewed as a complex concentrated in degree 0) and monoidal product given by the usual tensor product of complexes:

$$(V \otimes W)^n = \bigoplus_{p+q=n} V^p \otimes W^q,$$
$$d_{V\otimes W}(v \otimes w) = d_V(v) \otimes w + (-1)^{|v|} v \otimes d_W(w).$$

We will always employ the Koszul sign rule. Heuristically, this means that anytime we make two consecutive graded symbols s and t (with degrees $|s|$ and $|t|$) commute, we must multiply by $(-1)^{|s||t|}$:

$$st \rightsquigarrow (-1)^{|s||t|} ts. \tag{1.1.1}$$

In particular, the symmetry isomorphism is given by

$$V \otimes W \xrightarrow{\sim} W \otimes V,$$
$$v \otimes w \mapsto (-1)^{|v||w|} w \otimes v.$$

The monoidal category $\mathsf{C}(\mathbf{k})$ is closed. That is, we have a natural ($\mathbf{k}$-linear) bijection

$$\operatorname{Hom}(Z \otimes V, W) \xrightarrow{\sim} \operatorname{Hom}(Z, \underline{\operatorname{Hom}}(V, W)), \tag{1.1.2}$$

where the hom-complex $\underline{\mathrm{Hom}}(V,W)$ of maps from V to W is given by:

$$\begin{aligned} \underline{\mathrm{Hom}}(V,W)^n &= \mathrm{Hom}_{\mathrm{gr}}(V, W[n]), \\ df &= d_W \circ f - (-1)^{|f|} f \circ d_V. \end{aligned} \tag{1.1.3}$$

Here, $W[n]$ is the n-shift of the complex W, and $\mathrm{Hom}_{\mathrm{gr}}(V, W[n])$ is the module of morphisms $V \to W[n]$ in the category of graded $\mathbf{k}$-modules (that is, degree n morphisms $V \to W$). The above bijection (1.1.2) is given by

$$f \mapsto (z \mapsto f_z),$$

where $f_z(v) = f(z \otimes v)$. This bijection actually lifts to an isomorphism of complexes:

$$\underline{\mathrm{Hom}}(Z \otimes V, W) \xrightarrow{\sim} \underline{\mathrm{Hom}}(Z, \underline{\mathrm{Hom}}(V,W)). \tag{1.1.4}$$

The category of complexes can be used as a base for enrichment:

Definition 1.1.1. A *differential graded (dg-) category* is a category $\mathbf{A}$ enriched over $\mathsf{C}(\mathbf{k})$. That is, $\mathbf{A}$ is given by a set of objects $\mathrm{Ob}\,\mathbf{A}$, a hom-complex $\mathbf{A}(A,B)$ for any couple of objects A, B, and (unital, associative) composition maps:

$$\begin{aligned} \mathbf{A}(B,C) \otimes \mathbf{A}(A,B) &\to \mathbf{A}(A,C), \\ g \otimes f &\mapsto gf = g \circ f \end{aligned}$$

Remark 1.1.2. Giving a dg-category $\mathbf{A}$ is equivalent to giving a set of objects $\mathrm{Ob}\,\mathbf{A}$, a complex of $\mathbf{k}$-modules $\mathbf{A}(A,B)$ for any couple of objects A, B, and $\mathbf{k}$-bilinear composition maps

$$\mathbf{A}(B,C)^q \times \mathbf{A}(A,B)^p \to \mathbf{A}(A,C)^{p+q}$$

which are associative and unital, subject to the *graded Leibniz rule*:

$$d(fg) = (df)g + (-1)^{|f|} f(dg). \tag{1.1.5}$$

For any $A \in \mathrm{Ob}\,\mathbf{A}$, the identity 1_A is a closed degree 0 morphism: $1_A \in \mathbf{A}(A,A)^0$ and $d(1_A) = 0$.

Definition 1.1.3. Let $\mathbf{A}$ be a dg-category. The *opposite dg-category* $\mathbf{A}^{\mathrm{op}}$ is the dg-category with $\mathrm{Ob}\,\mathbf{A}^{\mathrm{op}} = \mathrm{Ob}\,\mathbf{A}$, $\mathbf{A}^{\mathrm{op}}(A,B) = \mathbf{A}(B,A)$, and compositions defined by

$$\begin{aligned} &\mathbf{A}^{\mathrm{op}}(B,C) \otimes \mathbf{A}^{\mathrm{op}}(A,B) \\ &= \mathbf{A}(C,B) \otimes \mathbf{A}(B,A) \\ &\cong \mathbf{A}(B,A) \otimes \mathbf{A}(C,B) \to \mathbf{A}(C,A) = \mathbf{A}^{\mathrm{op}}(A,C), \end{aligned}$$

using the symmetry isomorphism of the monoidal structure on $\mathsf{C}(\mathbf{k})$ and the composition in $\mathbf{A}$.

Remark 1.1.4. Given a dg-category $\mathbf{A}$ and $f \in \mathbf{A}(A, B)$, we write f^{op} for the corresponding morphism in $\mathbf{A}^{\mathrm{op}}(B, A)$. Then, the composition of two morphisms f^{op} and g^{op} in $\mathbf{A}^{\mathrm{op}}$ is spelled out as follows:

$$f^{\mathrm{op}} g^{\mathrm{op}} = (-1)^{|f||g|} (gf)^{\mathrm{op}},$$

whenever f and g are homogeneous of degrees $|f|$ and $|g|$.

Definition 1.1.5. Let $\mathbf{A}$ and $\mathbf{B}$ be two dg-categories. The *tensor product* $\mathbf{A} \otimes \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$ is the dg-category such that $\operatorname{Ob}(\mathbf{A} \otimes \mathbf{B}) = \operatorname{Ob} \mathbf{A} \times \operatorname{Ob} \mathbf{B}$, and

$$(\mathbf{A} \otimes \mathbf{B})((A, B), (A', B')) = \mathbf{A}(A, A') \otimes \mathbf{B}(B, B'),$$

with compositions given by tensor products of compositions, namely:

$$\begin{aligned}
&(\mathbf{A} \otimes \mathbf{B})((A', B'), (A'', B'')) \otimes (\mathbf{A} \otimes \mathbf{B})((A, B), (A', B')) \\
&\quad = \mathbf{A}(A', A'') \otimes \mathbf{B}(B', B'') \otimes \mathbf{A}(A, A') \otimes \mathbf{B}(B, B') \\
&\quad \cong (\mathbf{A}(A', A'') \otimes \mathbf{A}(A, A')) \otimes (\mathbf{B}(B', B'') \otimes \mathbf{B}(B, B')) \\
&\quad \to \mathbf{A}(A, A'') \otimes \mathbf{B}(B, B'') \\
&\qquad = (\mathbf{A} \otimes \mathbf{B})((A, B), (A'', B'')).
\end{aligned}$$

Remark 1.1.6. The composition of two morphisms $f \otimes g$ and $f' \otimes g'$ in $\mathbf{A} \otimes \mathbf{B}$ can be spelled out as follows:

$$(f' \otimes g')(f \otimes g) = (-1)^{|g'||f|} f' f \otimes g' g,$$

assuming g' and f are homogeneous. The identity morphism $1_{(A,B)}$ is clearly given by $1_A \otimes 1_B$.

Remark 1.1.7. The tensor product of dg-categories commutes with taking opposites:

$$(\mathbf{A} \otimes \mathbf{B})^{\mathrm{op}} = \mathbf{A}^{\mathrm{op}} \otimes \mathbf{B}^{\mathrm{op}}.$$

In fact, objects and hom-complexes are equal, and this is also true for compositions, thanks to monoidal coherence in $\mathsf{C}(\mathbf{k})$.

Example 1.1.8. The monoidal category of complexes $\mathsf{C}(\mathbf{k})$ is enriched over itself, with the hom-complexes defined in (1.1.3). We denote this dg-category by $\mathsf{C}_{\mathrm{dg}}(\mathbf{k})$.

Definition 1.1.9. Let $\mathbf{A}$ and $\mathbf{B}$ be dg-categories. A *dg-functor* $F \colon \mathbf{A} \to \mathbf{B}$ is a $\mathsf{C}(\mathbf{k})$-enriched functor of enriched categories. In other words, F consists of the following data:

- a function $F \colon \operatorname{Ob} \mathbf{A} \to \operatorname{Ob} \mathbf{B}$;
- for any couple of objects (A, B) of $\mathbf{A}$, a chain map

$$F = F_{(A,B)} \colon \mathbf{A}(A, B) \to \mathbf{B}(F(A), F(B)),$$

subject to the usual associativity and unitality axioms.

Remark 1.1.10. Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be dg-categories. A dg-functor $F\colon \mathbf{A} \otimes \mathbf{B} \to \mathbf{C}$ is called a *dg-bifunctor*. For any $A \in \mathbf{A}$ and $B \in \mathbf{B}$, F induces dg-functors $F(A,-)\colon \mathbf{B} \to \mathbf{C}$ and $F(-,B)\colon \mathbf{A} \to \mathbf{C}$, defined as follows: given homogeneous morphisms $f\colon A \to A'$ in $\mathbf{A}$ and $g\colon B \to B'$ in $\mathbf{B}$, we set

$$
\begin{aligned}
F(A,g) &= F(1_A \otimes g)\colon F(A,B) \to F(A,B'), \\
F(f,B) &= F(f \otimes 1_B)\colon F(A,B) \to F(A',B).
\end{aligned}
$$

Next, notice that

$$
\begin{aligned}
F(A',g)F(f,B) &= F(1_A \otimes g)F(f \otimes 1_B) = (-1)^{|f||g|}F(f \otimes g), \\
F(f,B')F(A,g) &= F(f \otimes 1_B)F(1_A \otimes g) = F(f \otimes g),
\end{aligned}
$$

hence the following diagram is commutative up to the sign $(-1)^{|f||g|}$, and the diagonal gives $F(f \otimes g)$:

$$
\begin{array}{ccc}
F(A,B) & \xrightarrow{F(f,B)} & F(A',B) \\
{\scriptstyle F(A,g)}\downarrow & \searrow^{F(f\otimes g)} & \downarrow{\scriptstyle F(A',g)} \\
F(A,B') & \xrightarrow[F(f,B')]{} & F(A',B').
\end{array}
$$

Conversely, assume that for all $A \in \mathbf{A}$ and $B \in \mathbf{B}$ we are given dg-functors $F_A\colon \mathbf{B} \to \mathbf{C}$ and $F_B\colon \mathbf{A} \to \mathbf{C}$, such that $F_A(B) = F_B(A) =: F(A,B)$ and the diagram

$$
\begin{array}{ccc}
F(A,B) & \xrightarrow{F_B(f)} & F(A',B) \\
{\scriptstyle F_A(g)}\downarrow & & \downarrow{\scriptstyle F_{A'}(g)} \\
F(A,B') & \xrightarrow[F_{B'}(f)]{} & F(A',B').
\end{array}
$$

is commutative up to the sign $(-1)^{|f||g|}$, whenever $f \in \mathbf{A}(A,A')$ and $g \in \mathbf{B}(B,B')$ are homogeneous morphisms. Then, there is a dg-bifunctor $F\colon \mathbf{A} \otimes \mathbf{B} \to \mathbf{C}$ which is defined on objects by $(A,B) \mapsto F(A,B)$, and on morphisms by the diagonal of the above diagram. F is the unique dg-bifunctor such that $F(-,B) = F_B$ and $F(A,-) = F_A$.

A dg-bifunctor $F\colon \mathbf{A} \otimes \mathbf{B} \to \mathbf{C}$ can be thought as a "dg-functor of two variables" $A \in \mathbf{A}$ and $B \in \mathbf{B}$, functorial in both A and B. The same is true in general for dg-functors $\mathbf{A}_1 \otimes \cdots \otimes \mathbf{A}_n \to \mathbf{C}$: they can be viewed as "dg-functors of many variables", with functoriality in each variable. Sometimes, we will employ *Einstein notation* to indicate which variables are covariant and which ones are contravariant. For example, a dg-functor $F\colon \mathbf{B}^{\mathrm{op}} \otimes \mathbf{A}_1 \otimes \mathbf{A}_2 \to \mathbf{C}$ will be written as

$$
F(B, A_1, A_2) = F^{B}_{A_1,A_2};
$$

lower variables are covariant, whereas upper variables are contravariant. Moreover, we shall set (for instance)

$$F^f_{A_1,A_2} = F(f \otimes 1_{A_1} \otimes 1_{A_2}),$$

for any morphism f.

Small dg-categories and dg-functors form a category, which is denoted by $\mathsf{dgCat}_{\mathbf{k}}$, or simply dgCat when the base ring is clear.

There are usual operations that can be performed on a complex V of $\mathbf{k}$-modules: taking the graded module of cocycles $Z^*(V)$ and the graded module of cohomology $H^*(V)$. Actually, these operations define functors:

$$Z^* \colon \mathsf{C}(\mathbf{k}) \to \mathsf{Gr}(\mathbf{k}), \tag{1.1.6}$$

$$H^* \colon \mathsf{C}(\mathbf{k}) \to \mathsf{Gr}(\mathbf{k}), \tag{1.1.7}$$

where $\mathsf{Gr}(\mathbf{k})$ denotes the monoidal category of graded $\mathbf{k}$-modules. Restricting ourselves to the zeroth degree, we also get functors

$$Z^0 \colon \mathsf{C}(\mathbf{k}) \to \mathsf{Mod}(\mathbf{k}), \tag{1.1.8}$$

$$H^0 \colon \mathsf{C}(\mathbf{k}) \to \mathsf{Mod}(\mathbf{k}), \tag{1.1.9}$$

to the monoidal category $\mathsf{Mod}(\mathbf{k})$ of $\mathbf{k}$-modules. All of these functors are *lax monoidal*: this means, taking H^* as an example, that we have natural morphisms involving the monoidal products and units:

$$\begin{aligned} H^*(V) \otimes H^*(W) &\to H^*(V \otimes W), \\ \mathbf{k} &\to H^*(\mathbf{k}). \end{aligned} \tag{1.1.10}$$

(The second one is actually an isomorphism in our cases.) Hence, we obtain induced functors at the level of enriched categories:

$$Z^* \colon \mathsf{dgCat} \to \mathsf{Gr}(\mathbf{k})\text{-}\mathsf{Cat}, \tag{1.1.11}$$

$$H^* \colon \mathsf{dgCat} \to \mathsf{Gr}(\mathbf{k})\text{-}\mathsf{Cat}, \tag{1.1.12}$$

$$Z^0 \colon \mathsf{dgCat} \to \mathbf{k}\text{-}\mathsf{Cat}, \tag{1.1.13}$$

$$H^0 \colon \mathsf{dgCat} \to \mathbf{k}\text{-}\mathsf{Cat}. \tag{1.1.14}$$

For instance, given a dg-category $\mathbf{A}$, the graded category $H^*(\mathbf{A})$ is defined as follows: $\operatorname{Ob} H^*(\mathbf{A}) = \operatorname{Ob} \mathbf{A}$, $H^*(\mathbf{A})(A,B) = H^*(\mathbf{A}(A,B))$, and compositions given by

$$H^*(\mathbf{A})(B,C) \otimes H^*(\mathbf{A})(A,B) \to H^*(\mathbf{A}(B,C) \otimes \mathbf{A}(A,C)) \to H^*(\mathbf{A}(A,C)),$$

where the first arrow comes from (1.1.10) and the second one is induced by composition in $\mathbf{A}$. For any dg-category $\mathbf{A}$, $Z^0(\mathbf{A})$ is called the *underlying category* of $\mathbf{A}$, and $H^0(\mathbf{A})$ is called the *homotopy category* of $\mathbf{A}$. There are natural projection functors:

$$Z^*(\mathbf{A}) \to H^*(\mathbf{A}),$$

$$Z^0(\mathbf{A}) \to H^0(\mathbf{A}).$$

The projection into cohomology will be often denoted by square parentheses, namely, if $f\colon A \to A'$ is a closed morphism in $\mathbf{A}$, then $[f]\colon A \to A'$ denotes its cohomology class, which is the corresponding morphism in $H^*(\mathbf{A})$.

We are now able to speak of isomorphisms and equivalences between objects in a dg-category. Given $A, A' \in \mathbf{A}$, we say that A *is dg-isomorphic to* A', and write $A \cong A'$, if A and A' are isomorphic in $Z^0(\mathbf{A})$; also, we say that A and A' are *homotopy equivalent*, and write $A \approx A'$, if A and A' are isomorphic in $H^0(\mathbf{A})$. A degree 0 morphism $f\colon A \to A'$ in $\mathbf{A}$ is called a *dg-isomorphism* if $Z^0(f)$ is an isomorphism in $Z^0(\mathbf{A})$; it is called a *homotopy equivalence* if $H^0(f)$ is an isomorphism in $H^0(\mathbf{A})$.

Remark 1.1.11. The functor $Z^0\colon \mathsf{C}(\mathbf{k}) \to \mathsf{Mod}(\mathbf{k})$ is essentially the *underlying functor* in the usual meaning within enriched category theory. indeed, for any complex V,

$$Z^0(V) = \operatorname{Hom}(\mathbf{k}, V)$$

as $\mathbf{k}$-modules.

Remark 1.1.12. The underlying category of the dg-category $\mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ is the category of complexes: $Z^0(\mathsf{C}_{\mathrm{dg}}(\mathbf{k})) = \mathsf{C}(\mathbf{k})$. Its homotopy category $\mathsf{K}(\mathbf{k}) = H^0(\mathsf{C}_{\mathrm{dg}}(\mathbf{k}))$ is the category of cochain complexes and homotopy classes of chain maps.

1.2 The dg-category of dg-functors

Let $\mathbf{A}, \mathbf{B}$ be dg-categories. Then, using the fact that the tensor product of complexes is symmetric, we can easily show that $\mathbf{A} \otimes \mathbf{B} \cong \mathbf{B} \otimes \mathbf{A}$ in dgCat. Hence, the tensor product of dg-categories endows dgCat with a structure of symmetric monoidal category; the monoidal unit is given by the dg-category $\mathbf{1} = \mathbf{1}_{\mathbf{k}}$, with a single object E_0 and $\mathbf{1}(E_0, E_0) \cong \mathbf{k}\langle 1_{E_0}\rangle$. We are going to show that dgCat is indeed a *closed* symmetric monoidal category.

Definition 1.2.1. Let $F, G\colon \mathbf{A} \to \mathbf{B}$ be dg-functors. A *dg-natural transformation* $\varphi\colon F \to G$ of degree p is a collection of degree p morphisms

$$\varphi_A\colon F(A) \to G(A),$$

for all $A \in \mathbf{A}$, such that for any degree q morphism $f \in \mathbf{A}(A, A')$ the following diagram is commutative up to the sign $(-1)^{|p||q|}$:

$$\begin{array}{ccc} F(A) & \xrightarrow{\varphi_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(A') & \xrightarrow{\varphi_{A'}} & G(A'). \end{array}$$

We write $\operatorname{Nat}_{\mathrm{dg}}(F, G)$ for the *complex of dg-natural transformations* $F \to G$. The coboundary of $\varphi\colon F \to G$ is defined objectwise:

$$(d\varphi)_A = d(\varphi_A),$$

for all $A \in \mathbf{A}$. Dg-natural transformations can be composed:

$$\begin{gathered}\mathrm{Nat}_{\mathrm{dg}}(F_2, F_3) \otimes \mathrm{Nat}_{\mathrm{dg}}(F_1, F_2) \to \mathrm{Nat}_{\mathrm{dg}}(F_1, F_3),\\ \psi \otimes \varphi \mapsto \psi\varphi,\end{gathered}$$

where $(\psi\varphi)_A = \psi_A \varphi_A$ for all $A \in \mathbf{A}$. In the end, we may define the *dg-category of dg-functors* $\mathrm{Fun}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})$ as the dg-category whose objects are dg-functors $\mathbf{A} \to \mathbf{B}$ and whose complexes of morphisms are given by the complexes of dg-natural transformations, with the above compositions. This dg-category serves as the internal hom of the symmetric monoidal category dgCat:

Proposition 1.2.2. *Given dg-categories* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$*, there is an isomorphism in* dgCat*:*

$$\mathrm{Fun}_{\mathrm{dg}}(\mathbf{A} \otimes \mathbf{B}, \mathbf{C}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{dg}}(\mathbf{A}, \mathrm{Fun}_{\mathrm{dg}}(\mathbf{B}, \mathbf{C})), \tag{1.2.1}$$

natural in $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$*.*

Idea of proof. Given a dg-functor $F \colon \mathbf{A} \otimes \mathbf{B} \to C$, we define a dg-functor $F' \colon \mathbf{A} \to \mathrm{Fun}_{\mathrm{dg}}(\mathbf{B}, \mathbf{C})$ by

$$F'(A)(B) = F(A, B), \tag{1.2.2}$$

Vice-versa, starting from F', the above formula defines its image F. The definition at the level of dg-natural transformations is similar. □

Remark 1.2.3. Let $F, G \colon \mathbf{A} \otimes \mathbf{B} \to \mathbf{C}$ be dg-bifunctors. Then, it is easily proved that a family of maps $\varphi_{A,B} \colon F(A, B) \to G(A, B)$ gives a dg-natural transformation of dg-functors $F \to G$ if and only if it is "natural in both variables", that is, if and only if $\varphi_{-,B} \colon F(-, B) \to G(-, B)$ and $\varphi_{A,-} \colon F(A, -) \to G(A, -)$ are both dg-natural transformations for all $A \in \mathbf{A}$ and $B \in \mathbf{B}$. More generally, given "dg-functors of many variables" $F, G \colon A_1 \otimes \cdots \otimes A_n \to \mathbf{C}$, a family of maps $\varphi_{A_1,\dots,A_n} \colon F(A_1, \dots, A_n) \to G(A_1, \dots, A_n)$ gives a dg-natural trasfomation $F \to G$ if and only if it is "natural in each variable", in the sense explained above.

Remark 1.2.4. A (properly said) *natural transformation* φ of dg-functors $F, G \colon \mathbf{A} \to \mathbf{B}$ is by definition an element $\varphi \in Z^0(\mathrm{Nat}_{\mathrm{dg}}(F, G))$, that is, a *closed and degree* 0 dg-natural transformation; it is a morphism in the underlying category $Z^0(\mathrm{Fun}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B}))$. So, F and G are dg-isomorphic if and only if there exists a natural transformation $\varphi \colon F \to G$ such that $\varphi_A \colon F(A) \to G(A)$ is an isomorphism for all A.

Dg-functors with values in the dg-category of complexes are called *dg-modules* and are worth being studied in their own right.

Definition 1.2.5. Let $\mathbf{A}$ be a dg-category. A *left* $\mathbf{A}$*-dg-module* is a dg-functor $\mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$. A *right* $\mathbf{A}$*-dg-module* is a dg-functor $\mathbf{A}^{\mathrm{op}} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$.

Let $\mathbf{B}$ be another dg-category. A $\mathbf{A}$-$\mathbf{B}$*-dg-bimodule* is a dg-bifunctor $\mathbf{B}^{\mathrm{op}} \otimes \mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$.

Remark 1.2.6. Let $F\colon \mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ be a left dg-module. Given a couple of objects (A, A'), we may view the functor F on the hom-complex $\mathbf{A}(A, A')$ as an element

$$\begin{aligned} F_{(A,A')} &\in \operatorname{Hom}(\mathbf{A}(A,A'), \underline{\operatorname{Hom}}(F(A), F(A'))) \\ &\cong \operatorname{Hom}(\mathbf{A}(A,A') \otimes F(A), F(A')). \end{aligned}$$

So, giving F as a functor $\mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ is the same as giving a complex of $\mathbf{k}$-modules $F(A)$ for all objects $A \in \mathbf{A}$, and chain maps

$$\begin{aligned} \mathbf{A}(A,A') \otimes F(A) &\to F(A'), \\ f \otimes x &\mapsto fx, \end{aligned}$$

such that

$$\begin{aligned} g(fx) &= (gf)x, \\ 1_A x &= x, \end{aligned}$$

for any $x \in F(A)$, for any $f\colon A \to A'$ and $g\colon A' \to A''$. So, a left dg-module is given by a family of complexes parametrised by objects of $\mathbf{A}$, together with a left $\mathbf{A}$-action. By construction, we have

$$fx = F(f)(x).$$

Similarly, we have the characterisation of a right dg-module $F\colon \mathbf{A}^{\mathrm{op}} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ as family of complexes with a right action:

$$\begin{aligned} F(A) \otimes \mathbf{A}(A', A) &\to F(A'), \\ x \otimes f &\mapsto xf; \end{aligned}$$

in terms of F, we have

$$xf = (-1)^{|x||f|} F(f)(x),$$

taking into account the Kozsul sign rule. Finally, recalling Remark 1.1.10, giving a dg-bimodule $F\colon \mathbf{B}^{\mathrm{op}} \otimes \mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ is the same as giving a family of complexes $F(B, A)$ together with a left action of $\mathbf{A}$ and a right action of $\mathbf{B}$, subject to the compatibility condition:

$$(gx)f = g(xf),$$

whenever $x \in F(B, A)$ and $f\colon B' \to B, g\colon A \to A'$. We allow ourselves to drop parentheses and write gxf meaning $(gx)f = g(xf)$. Actually, in terms of the original bifunctor F, we have

$$F(f \otimes g)(x) = (-1)^{|f|(|x|+|g|)} gxf.$$

In the following, we will allow ourselves to shift freely from one characterisation of dg-(bi)modules to another, and adopt indiscriminately either the "functor" notation or the "left/right action" notation, keeping in mind how to interchange them.

The dg-category of right $\mathbf{A}$-modules is the category of functors $\mathrm{Fun}_{\mathrm{dg}}(\mathbf{A}^{\mathrm{op}}, \mathsf{C}_{\mathrm{dg}}(\mathbf{k}))$, and will be denoted by $\mathsf{C}_{\mathrm{dg}}(\mathbf{A})$. Moreover, we set:

$$\mathsf{C}(\mathbf{A}) = Z^0(\mathsf{C}_{\mathrm{dg}}(\mathbf{A})), \tag{1.2.3}$$

$$\mathsf{K}(\mathbf{A}) = H^0(\mathsf{C}_{\mathrm{dg}}(\mathbf{A})). \tag{1.2.4}$$

A morphism of left $\mathbf{A}$-modules $\varphi\colon F \to G$ is simply a dg-natural transformation of functors. Adopting the "left action" notation as explained in Remark 1.2.6, we see that φ can be viewed as a family of maps $\varphi_A\colon F(A) \to G(A)$ such that

$$\varphi_{A'}(fx) = (-1)^{|\varphi||f|} f\varphi_A(x),$$

for any $x \in F(A)$ and $f \in \mathbf{A}(A, A')$ (notice the Koszul sign rule). Similarly, a morphism of right $\mathbf{A}$-modules $\psi\colon M \to N$ satisfies the following:

$$\psi_A(xf) = \psi_{A'}(x)f,$$

for any $x \in M(A')$ and $f \in \mathbf{A}(A, A')$. Finally, a morphism of $\mathbf{A}$-$\mathbf{B}$-bimodules $\xi\colon F_1 \to F_2$ is required to satisfy both compatibilities with the left and right actions:

$$\xi_{(B',A')}(gxf) = (-1)^{|g||\xi|} g\xi_{(B,A)}(x)f,$$

whenever $x \in F_1(B, A), f \in \mathbf{B}(B', B), g \in \mathbf{A}(A, A')$.

Yoneda lemma and Yoneda embedding

Let $\mathbf{A}$ be a dg-category. We associate to $\mathbf{A}$ an $\mathbf{A}$-$\mathbf{A}$-dg-bimodule called the *diagonal bimodule* and denoted by $h_{\mathbf{A}} = h$. It is defined by

$$h_{\mathbf{A}}(A, A') = h^A_{A'} = \mathbf{A}(A, A'), \tag{1.2.5}$$

with right and left actions given by composition in $\mathbf{A}$. Also, given a dg-functor $F\colon \mathbf{C} \to \mathbf{A}$, we denote respectively by h^F and h_F the $\mathbf{A}$-$\mathbf{C}$-dg-bimodule and the $\mathbf{C}$-$\mathbf{A}$-bimodule defined by:

$$\begin{aligned} h^F(C, A) &= h^{F(C)}_A = \mathbf{A}(F(C), A), \\ h_F(A, C) &= h^A_{F(C)} = \mathbf{A}(A, F(C)). \end{aligned} \tag{1.2.6}$$

The left and right actions of $\mathbf{A}$ and $\mathbf{C}$ on h^F are defined by the following compositions in $\mathbf{A}$:

$$\begin{aligned} gf &= g \circ F(f), \\ g'g &= g' \circ g, \end{aligned}$$

whenever $f \in \mathbf{C}(C', C)$, $g \in \mathbf{A}(F(C), A)$ and $g' \in \mathbf{A}(A, A')$. The actions on h_F are defined analogously. Moreover, if $G\colon \mathbf{B} \to \mathbf{A}$ is another dg-functor, there is a $\mathbf{B}$-$\mathbf{C}$-dg-bimodule h^F_G defined by:

$$h^F_G(C, B) = \mathbf{A}(F(C), G(B)), \tag{1.2.7}$$

with left and right actions defined in a similar way as above.

Taking the components of the diagonal bimodule, we obtain the right dg-modules $h_A = \mathbf{A}(-, A)$ and the left dg-modules $h^A = \mathbf{A}(A, -)$. A right (resp. left) $\mathbf{A}$-dg-module F is said to be *representable* if $F \cong h_A$ (resp. $F \cong h^A$) for some $A \in \mathbf{A}$. The well-known Yoneda lemma has a counterpart in the differential graded framework:

Theorem 1.2.7 (Dg-Yoneda lemma)**.** *Let $F \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A})$ be a right $\mathbf{A}$-dg-module, and let $A \in \mathbf{A}$. Then, there is an isomorphism of complexes:*

$$\begin{aligned} \mathrm{Nat}_{\mathrm{dg}}(h_A, F) &\xrightarrow{\sim} F(A), \\ \varphi &\mapsto \varphi_A(1_A), \end{aligned} \tag{1.2.8}$$

natural both in A and F.

Proof. The above map is clearly a chain map, and naturality in A and F is checked directly. We may conclude by showing that it is bijective. Let $x \in F(A)$. We look for a dg-natural transformation $\varphi \colon h_A \to F$ such that $\varphi_A(1_A) = x$. If $f \in \mathbf{A}(A', A) = h_A^{A'}$, then we must have that

$$\varphi_{A'}(f) = \varphi_{A'}(1_A f) = \varphi_A(1_A) f = xf, \tag{1.2.9}$$

so we see that φ can be defined in a unique way. To conclude, we just have to show that the above definition actually gives a dg-natural transformation, but this is straightforward. □

Remark 1.2.8. Assume that a right $\mathbf{A}$-dg-module F is representable: there exists an isomorphism $\varphi \colon h_A \xrightarrow{\sim} F$ for some $A \in \mathbf{A}$. Yoneda lemma (and its proof) tells us that this isomorphism is of the form $\varphi_{A'}(f) = ef$ for some (uniquely determined) $e \in Z^0(F(A))$. The bijectivity of $\varphi_{A'}$ implies that for any $y \in F(A')$ there exists a unique $f \colon A' \to A$ such that $y = ef$. This actually characterises representable $\mathbf{A}$-modules: $F \cong h_A$ if and only if there exists an element $e \in F(A)$, closed and of degree 0, such that for any $y \in F(A')$ there exists a unique $f \colon A' \to A$ such that $y = ef$. The reader should compare this discussion with the definition of universal arrow and the notion of counit of an adjunction.

Remark 1.2.9. Taking $\mathbf{A}^{\mathrm{op}}$ instead of $\mathbf{A}$ in the above discussion gives the Yoneda lemma for left $\mathbf{A}$-modules, that is, the following natural isomorphism of complexes:

$$\begin{aligned} \mathrm{Nat}_{\mathrm{dg}}(h^A, F) &\xrightarrow{\sim} F(A), \\ \varphi &\mapsto \varphi_A(1_A), \end{aligned}$$

whenever $F \colon \mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ is a left $\mathbf{A}$-module, and $A \in \mathbf{A}$. Also, Remark 1.2.8 can be dualised, giving the following characterisation of representable left $\mathbf{A}$-modules: $F \cong h^A$ if and only if there exist an element $n \in Z^0(F(A))$ such that for any $y \in F(A')$ there exists a unique $f \colon A \to A'$ such that $y = fn$.

Now, let us come back to the diagonal bimodule $h = h_{\mathbf{A}}$. As we said, it is a dg-functor $h\colon \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$. By (1.2.1) (also using that $\mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \cong \mathbf{A} \otimes \mathbf{A}^{\mathrm{op}}$), it can be viewed as a dg-functor (we abuse notation)

$$\begin{aligned} h\colon \mathbf{A} &\to \mathsf{C}_{\mathrm{dg}}(\mathbf{A}), \\ A &\mapsto h_A, \\ f &\mapsto h_f\colon h_{A_1} \to h_{A_2}, \end{aligned} \tag{1.2.10}$$

whenever $f \in \mathbf{A}(A_1, A_2)$, where by definition

$$h_f^A(g) = fg,$$

for all $g \in h_{A_1}^A$. So, we see that the map induced by h on morphisms

$$\mathbf{A}(A_1, A_2) \to \mathrm{Nat}_{\mathrm{dg}}(h_{A_1}, h_{A_2})$$

is actually the inverse of the Yoneda isomorphism (1.2.8). We conclude that the dg-functor (1.2.10) is fully faithful. It is called the *Yoneda embedding* of $\mathbf{A}$.

Dg-adjunctions

The notion of adjoint dg-functors is a direct generalisation of the notion of ordinary adjoint functors. In the following, we briefly recall the main results of the theory, interpreted in the differential graded framework.

Definition 1.2.10. Let $F\colon \mathbf{A} \leftrightarrows \mathbf{B}\colon G$ be dg-functors. We say that F is a *left adjoint of* G (and G is a *right adjoint* of F), writing $F \dashv G$, if there is an isomorphism of complexes:

$$\varphi_{A,B}\colon \mathbf{B}(F(A), B) \xrightarrow{\sim} \mathbf{A}(A, G(B)), \tag{1.2.11}$$

natural in both A and B.

Remark 1.2.11. Naturality can be expressed in terms of left and right actions. So, we have:

$$\begin{aligned} \varphi(f \circ F(a)) &= \varphi(f) \circ a, \\ \varphi(b \circ f) &= (-1)^{|b||f|} G(b) \circ \varphi(f), \end{aligned}$$

whenever $f\colon F(A) \to B$, $a\colon A' \to A$ and $b\colon B \to B'$.

The natural isomorphism φ can be viewed as an isomorphism of left $\mathbf{B}$-modules:

$$\varphi\colon h^{F(A)} \xrightarrow{\sim} h_G^A,$$

natural in $A \in \mathbf{A}$. Applying the Yoneda lemma, we find out that φ is determined by closed degree zero maps $\eta_A\colon A \to GF(A)$, such that for any $f\colon A \to G(B)$ there exists a unique $f' \colon F(A) \to B$ such that $f = G(f')\eta_A$:

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & G(B) \\ {\scriptstyle \eta_A}\big\downarrow & \overset{G(f')}{\nearrow} & \\ GF(A). & & \end{array} \tag{1.2.12}$$

Naturality in A implies that $\eta = (\eta_A) \colon 1_{\mathbf{A}} \to GF$ is a (closed, degree zero) natural transformation, which is called the *unit* of the adjunction. Similarly, we may view φ^{-1} as an isomorphism of right $\mathbf{A}$-modules:

$$\varphi^{-1} \colon h_{G(B)} \to h^F_B,$$

natural in $B \in \mathbf{B}$. So, the adjunction is also determined by the *counit* $\varepsilon \colon FG \to 1_{\mathbf{B}}$, a (closed, degree zero) natural transformation which satisfies the dual universal property:

$$\begin{array}{ccc} B & \xleftarrow{\quad g \quad} & F(A) \\ {\scriptstyle \varepsilon_B}\big\uparrow & \overset{F(g')}{\swarrow} & \\ FG(B). & & \end{array} \tag{1.2.13}$$

Clearly, a dg-adjunction $F \dashv G$ induces adjunctions $Z^0(F) \dashv Z^0(G)$ and $H^0(F) \dashv H^0(G)$, with units and counits naturally induced by the unit and counit of $F \dashv G$.

Remark 1.2.12. Adjoints are obtained just by checking universal properties. That is, assume we are given a dg-functor $G \colon \mathbf{B} \to \mathbf{A}$, and assume that for any $A \in \mathbf{A}$ we find an object $F(A) \in \mathbf{B}$ together with a closed degree 0 map $\eta_A \colon A \to GF(A)$ which satisfy the universal property (1.2.12); then, there is a unique way to define a dg-functor $F \colon \mathbf{A} \to \mathbf{B}$ such that $F \dashv G$ and η_A is the unit of the adjunction. To see this, just start from $f \colon A \to A'$ and apply (1.2.12) to $\eta_{A'} \circ f$. We find a unique morphism $F(f) \colon F(A) \to F(A')$ such that $GF(f)\eta_A = \eta_{A'} f$. Thanks to uniqueness, we are able to prove that $f \mapsto F(f)$ is a chain map, and compositions and units are preserved.

The dual statement is clearly true: if we start with $F \colon \mathbf{A} \to \mathbf{B}$ and we find $G(B) \in \mathbf{A}$ and $\varepsilon_B \colon FG(B) \to B$ which satisfy the dual universal property (1.2.13), then we get $G \colon \mathbf{B} \to \mathbf{A}$ such that $F \dashv G$ with counit given by the ε_B.

By definition, a dg-functor $F \colon \mathbf{A} \to \mathbf{B}$ is *(dg-)fully faithful* if for any $A, A' \in \mathbf{A}$, the map on hom-complexes

$$F_{(A,A')} \colon \mathbf{A}(A, A') \to \mathbf{B}(F(A), F(A'))$$

is an isomorphism of complexes. Moreover, we say that F is *(dg-)essentially surjective* if for any $B \in \mathbf{B}$ there exists $A \in \mathbf{A}$ such that $B \cong F(A)$ (in other words, $Z^0(F)$ is essentially surjective). When we are given an adjunction $F \dashv G$, then we have the following useful characterisation:

Proposition 1.2.13. *Let $F \dashv G \colon \mathbf{A} \leftrightarrows \mathbf{B}$ be an adjunction of dg-functors. Then, F is fully faithful if and only if the unit $\eta \colon 1_{\mathbf{A}} \to GF$ is an isomorphism. Dually, G is fully faithful if and only if the counit $\varepsilon \colon FG \to 1_{\mathbf{B}}$ is an isomorphism.*

This proposition can be strenghtened, giving a very useful result. It is a direct adaptation of [JM89, Lemma 1.3], which is proved using the formalism of (co)monads:

Proposition 1.2.14. *Let $F \dashv G \colon \mathbf{A} \leftrightarrows \mathbf{B}$ be an adjunction of dg-functors. Then, $GF \cong 1_{\mathbf{A}}$ if and only if the unit $\eta \colon 1_{\mathbf{A}} \to GF$ is an isomorphism. Dually, $FG \cong 1_{\mathbf{B}}$ if and only if the counit $\varepsilon \colon FG \to 1_{\mathbf{B}}$ is an isomorphism. In particular, F is fully faithful if and only if $GF \cong 1_{\mathbf{A}}$, and G is fully faithful if and only if $FG \cong 1_{\mathbf{B}}$.*

1.3 Quasi-equivalences and homotopy

There is a natural notion of equivalence between dg-categories. By definition, a dg-functor $F \colon \mathbf{A} \to \mathbf{B}$ is a *dg-equivalence* if there exists a dg-functor $F' \colon \mathbf{B} \to \mathbf{A}$ such that $F'F \cong 1_{\mathbf{A}}$ and $FF' \cong 1_{\mathbf{B}}$. Dg-equivalences have a similar characterisation as ordinary equivalences of categories:

Proposition 1.3.1. *Let $F \colon \mathbf{A} \to \mathbf{B}$ be a dg-functor. Then, F is a dg-equivalence if and only if it is fully faithful and essentially surjective.*

Fully faithfulness and essential surjectivity have "homotopy counterparts":

Definition 1.3.2. Let $F \colon \mathbf{A} \to \mathbf{B}$ be a dg-functor. We say that F is *quasi-fully faithful* if for any couple of objects $A, B \in \mathbf{A}$ the chain map

$$F_{(A,B)} \colon \mathbf{A}(A, B) \to \mathbf{B}(F(A), F(B))$$

is a quasi-isomorphism of complexes (that is, it induces an isomorphism in the graded cohomology). Moreover, we say that F is *quasi-essentially surjective* if $H^0(F)$ is essentially surjective. F is called a *quasi-equivalence* if it is both quasi-fully faithful and quasi-essentially surjective.

One could naively expect that a quasi-equivalence $F \colon \mathbf{A} \to \mathbf{B}$ has a "homotopy inverse dg-functor", that is, a dg-functor $F' \colon \mathbf{B} \to \mathbf{A}$ such that $F'F \approx 1_{\mathbf{A}}$ and $FF' \approx 1_{\mathbf{B}}$. Unfortunately, this is not the case. In general, if we define $\mathbf{A} \sim \mathbf{B}$ by "there exists a quasi-equivalence $\mathbf{A} \to \mathbf{B}$", we find a reflexive and transitive relation, but not a symmetric one. Given $\mathbf{A}$ and $\mathbf{B}$, we say that they are *quasi-equivalent*, writing $\mathbf{A} \overset{\text{qe}}{\approx} \mathbf{B}$, if there exists a zig-zag of quasi-equivalences:

$$\mathbf{A} \leftarrow \mathbf{A}_1 \to \ldots \leftarrow \mathbf{A}_n \to \mathbf{B}.$$

Understading dg-categories up to quasi-equivalence is one of the goals of the *homotopy theory of dg-categories.* The task is easier when we notice that the category dgCat has a natural model structure whose weak equivalences are the quasi-equivalences. In the following, we show the main features of this structure.

Definition 1.3.3. Let $F\colon \mathbf{A} \to \mathbf{B}$ be a dg-functor. We say that F is a *fibration* (or *isofibration*) if it satisfies the following two properties:

1. For any $A, A' \in \mathbf{A}$, the map
$$F_{(A,A')}\colon \mathbf{A}(A,A') \to \mathbf{B}(F(A),F(A'))$$
is surjective.

2. For any isomorphism $\overline{g}\colon B_0 \to B_1$ in $H^0(\mathbf{B})$ and any $A_1 \in F^{-1}(B_1)$, there exists an isomorphism $\overline{f}\colon A_0 \to A_1$ in $H^0(\mathbf{A})$ such that $H^0(F)(\overline{f}) = \overline{g}$.

Remark 1.3.4. Call a dg-functor a *trivial fibration* if it is both a fibration and a quasi-equivalence. If $G\colon \mathbf{A} \to \mathbf{B}$ is a trivial fibration, then G is strictly surjective on objects. In fact, given $B \in \mathbf{B}$, there is an isomorphism $B \xrightarrow{\sim} G(A)$ in $H^0(\mathbf{B})$ (G is a quasi-equivalence), and moreover there exists an isomorphism $A' \xrightarrow{\sim} A$ in $H^0(\mathbf{A})$ which is mapped by G in $B \xrightarrow{\sim} G(A)$ (G is a fibration); in particular, $G(A') = B$.

Notice that every dg-category $\mathbf{A}$ is *fibrant*, in the sense that the unique dg-functor $\mathbf{A} \to \mathbf{0}$ is a fibration ($\mathbf{0}$ is the dg-category with one object and trivial complex of morphisms).

Theorem 1.3.5 ([Tab05])**.** *The category* dgCat *of small dg-categories admits a model category structure, with weak equivalences given by quasi-equivalences and fibrations given by the isofibrations of Definition 1.3.3.*

Remark 1.3.6. Cofibrations in dgCat are defined to be the dg-functors which satisfy the left lifting property with respect to trivial fibrations. In particular, we see that a dg-category $\mathbf{C}$ is cofibrant if and only if any dg-functor $F\colon \mathbf{C} \to \mathbf{B}$ admits a lift along any trivial fibration $G\colon \mathbf{A} \to \mathbf{B}$:

$$\begin{array}{ccc} & & \mathbf{A} \\ & \overset{\widetilde{F}}{\nearrow} & \downarrow G \\ \mathbf{C} & \xrightarrow{F} & \mathbf{B}. \end{array} \tag{1.3.1}$$

We call Hqe the localisation of dgCat along quasi-equivalences, and denote by
$$\ell\colon \mathsf{dgCat} \to \mathsf{Hqe}$$
the localisation functor. Hqe is often called the *homotopy category of dg-categories*. The machinery of model categories allows us to describe morphisms in Hqe:

Proposition 1.3.7 (an application of [Hov99, Theorem 1.2.10])**.** *A morphism* $\widehat{F}\colon \mathbf{A} \to \mathbf{B}$ *in* Hqe *is represented by a roof:*
$$\mathbf{A} \xleftarrow{G} Q(\mathbf{A}) \xrightarrow{F} \mathbf{B},$$
in the sense that $\widehat{F} = \ell(F)\ell(G)^{-1}$*. Moreover,* G *is a trivial fibration, and* $Q(\mathbf{A})$ *is a cofibrant dg-category (often called a* cofibrant replacement *of* $\mathbf{A}$*).*

Remark 1.3.8. A morphism $\widehat{F} \in \mathsf{Hqe}(\mathbf{A}, \mathbf{B})$, described as above, clearly yields an ordinary functor $H^0(\widehat{F})\colon H^0(\mathbf{A}) \to H^0(\mathbf{B})$, defined up to isomorphism. This procedure will be made clearer in the following chapters.

Remark 1.3.9. If the domain dg-category $\mathbf{A}$ is already cofibrant, then a morphism $\widehat{F}\colon \mathbf{A} \to \mathbf{B}$ in Hqe is represented by a dg-functor $F\colon \mathbf{A} \to \mathbf{B}$, in the sense that $\widehat{F} = \ell(F)$. A main problem is understanding what it means that $\ell(F) = \ell(G)$, that is, when two dg-functors induce the same morphism in **Hqe**.

The homotopy category **Hqe** has a structure of symmetric monoidal category. Given two dg-categories $\mathbf{A}$ and $\mathbf{B}$, its *derived tensor product* is defined (up to quasi-equivalence) as:

$$\mathbf{A} \otimes^{\mathbb{L}} \mathbf{B} = \mathbf{A}^{\mathrm{hp}} \otimes \mathbf{B} \overset{\mathrm{qe}}{\approx} \mathbf{A} \otimes \mathbf{B}^{\mathrm{hp}} \overset{\mathrm{qe}}{\approx} \mathbf{A}^{\mathrm{hp}} \otimes \mathbf{B}^{\mathrm{hp}}, \tag{1.3.2}$$

where $\mathbf{A}^{\mathrm{hp}}$ is a *h-projective resolution* of $\mathbf{A}$ (see, for instance, [CS15, Remark 2.7]). An important result is that this monoidal structure is closed:

Theorem 1.3.10 ([Toë07]). *For any pair of dg-categories* $\mathbf{A}$ *and* $\mathbf{B}$*, there is a dg-category* $\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B})$ *such that there is a bijection*

$$\mathsf{Hqe}(\mathbf{A} \otimes^{\mathbb{L}} \mathbf{B}, \mathbf{C}) \xrightarrow{\sim} \mathsf{Hqe}(\mathbf{A}, \mathbb{R}\underline{\mathrm{Hom}}(\mathbf{B}, \mathbf{C})), \tag{1.3.3}$$

natural in $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{Hqe}$*. The above bijection lifts to a natural quasi-equivalence*

$$\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A} \otimes^{\mathbb{L}} \mathbf{B}, \mathbf{C}) \xrightarrow{\sim} \mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbb{R}\underline{\mathrm{Hom}}(\mathbf{B}, \mathbf{C})). \tag{1.3.4}$$

Moreover, the set of morphisms $\mathsf{Hqe}(\mathbf{A}, \mathbf{B})$ *is in natural bijection with the isomorphism classes of* $H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B}))$.

The internal hom is defined up to quasi-equivalence, and has many "incarnations", which we will see in the following chapters. Normally, we will allow ourselves to shift from one description to another, keeping the $\mathbb{R}\underline{\mathrm{Hom}}$ notation.

A very useful tool in computations within the homotopy category Hqe is given by *homotopies.* In the model category dgCat, right homotopies are given by "homotopy coherent, homotopy invertible natural transformations" between dg-functors. To describe them, we first need to define the dg-category of "homotopy coherent morphisms" of a given dg-category:

Definition 1.3.11. Let $\mathbf{A}$ be a dg-category. The *dg-category of (homotopy coherent) morphisms* $\underline{\mathrm{Mor}}\,\mathbf{A}$ is defined as follows. Objects are triples (A, B, f), where $f \in Z^0(\mathbf{A}(A, B))$. A degree n morphism $(A, B, f) \to (A', B', f')$ is given by a lower triangular matrix

$$(u, v, h) = \begin{pmatrix} u & 0 \\ h & v \end{pmatrix},$$

where $u \in \mathbf{A}(A, A')^n$, $v \in \mathbf{A}(B, B')^n$ and $h \in \mathbf{A}(A, B)^{n-1}$. Compositions are defined by matrix multiplication with a sign rule:

$$\begin{pmatrix} u' & 0 \\ h' & v' \end{pmatrix} \begin{pmatrix} u & 0 \\ h & v \end{pmatrix} = \begin{pmatrix} u'u & 0 \\ (-1)^n h'u + v'h & v'v \end{pmatrix},$$

whenever (u, v, h) has degree n. The differential of a morphism $(u, v, h)\colon (A, B, f) \to (A', B', f')$ of degree n is defined by

$$d\begin{pmatrix} u & 0 \\ h & v \end{pmatrix} = \begin{pmatrix} du & 0 \\ dh + (-1)^n(f'u - vf) & dv \end{pmatrix}.$$

There are obvious "source" and "target" dg-functors:

$$\begin{aligned} S\colon \underline{\mathrm{Mor}}\,\mathbf{A} \to \mathbf{A}, \quad (A, B, f) \mapsto A, \quad (u, v, h) \mapsto u, \\ T\colon \underline{\mathrm{Mor}}\,\mathbf{A} \to \mathbf{A}, \quad (A, B, f) \mapsto B, \quad (u, v, h) \mapsto v. \end{aligned}$$

Notice that the chosen sign conventions in the definition of $\underline{\mathrm{Mor}}\,\mathbf{A}$ allow to define S and T in the simplest way, without any sign twist. The dg-category $\underline{\mathrm{Mor}}\,\mathbf{A}$ has a full subcategory $P(\mathbf{A})$ whose objects are the morphisms (A, B, f) such that f is a homotopy equivalence. Now, we may define the notion of right homotopy of dg-functors:

Definition 1.3.12. Let $F, G\colon \mathbf{A} \to \mathbf{B}$ be dg-functors. A *right homotopy* from F to G is a dg-functor $\varphi\colon \mathbf{A} \to P(\mathbf{B})$ such that the diagram

$$\begin{array}{ccc} & & \mathbf{B} \\ & {\scriptstyle F}\nearrow & \uparrow{\scriptstyle S} \\ \mathbf{A} & \xrightarrow{\varphi} & P(\mathbf{B}) \\ & {\scriptstyle G}\searrow & \downarrow{\scriptstyle T} \\ & & \mathbf{B} \end{array} \tag{1.3.5}$$

is commutative. In this case, we will say that F and G are *right homotopic*.

The dg-category $P(\mathbf{B})$ defines a *path object* for $\mathbf{B}$ in the model category dgCat (see [Tab10, Proposition 3.3]). Hence, the general theory of model categories, in particular [Hov99, Theorem 1.2.10], gives us the following result, which is an important tool for concrete computations:

Corollary 1.3.13. *Let $\mathbf{A}$ and $\mathbf{B}$ be dg-categories, and assume that $\mathbf{A}$ is cofibrant. Let $\widehat{F}, \widehat{G} \in \mathsf{Hqe}(\mathbf{A}, \mathbf{B})$. Write $\widehat{F} = \ell(F)$ and $\widehat{G} = \ell(G)$ for some dg-functors $F, G\colon \mathbf{A} \to \mathbf{B}$. Then, $\widehat{F} = \widehat{G}$ if and only if F and G are right homotopic.*

Remark 1.3.14. Let $\mathbf{B}$ be a dg-category. There is a natural functor

$$\begin{aligned} H^0(\underline{\mathrm{Mor}}\,\mathbf{B}) &\to \mathrm{Mor}\, H^0(\mathbf{B}), \\ (A, B, f) &\mapsto (A, B, [f]), \\ [(u, v, h)] &\mapsto ([u], [v]), \end{aligned} \tag{1.3.6}$$

where $\mathrm{Mor}\, H^0(\mathbf{B})$ denotes the ordinary category of morphisms of $H^0(\mathbf{B})$. It will be proven that this functor is surjective, full, and reflects isomorphisms (see Proposition 4.2.2); we think of it as the functor which "forgets homotopies". Now, let $F, G\colon \mathbf{A} \to \mathbf{B}$

be dg-functors, and let $\varphi\colon \mathbf{A} \to \underline{\mathrm{Mor}}\, \mathbf{B}$ be a dg-functor such that $S\varphi = F$ and $T\varphi = G$: we call it a *directed homotopy*. Taking H^0 and composing with (1.3.6), we obtain a functor, which abusing notation we denote

$$H^0(\varphi)\colon H^0(\mathbf{A}) \to \mathrm{Mor}\, H^0(\mathbf{B}).$$

This functor satisfies $S \circ H^0(\varphi) = H^0(F)$ and $T \circ H^0(\varphi) = H^0(G)$, where S and T are, again abusing notation, the source and target functors $\mathrm{Mor}\, H^0(\mathbf{B}) \to H^0(\mathbf{B})$. Now, it is well known that such a functor corresponds to a natural transformation

$$H^0(\varphi)\colon H^0(F) \to H^0(G).$$

So, we see that a directed homotopy $\varphi\colon F \to G$ as above yields a natural transformation $H^0(F) \to H^0(G)$. Notice, moreover, that φ is a right homotopy if (and only if) $H^0(\varphi)$ is a natural isomorphism.

The author is pretty convinced that directed homotopies, in the case that $\mathbf{A}$ is cofibrant, are to be identified with morphisms $F \to G$ in $Z^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B}))$: since $\mathbf{A}$ is cofibrant, it seems to be ok to assume that objects of $\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B})$ are dg-functors. The above procedure should yield a functor

$$Z^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B})) \to \mathrm{Fun}(H^0(\mathbf{A}), H^0(\mathbf{B})),$$

which should induce (an incarnation of) the functor $\Phi^{\mathbf{A}\to\mathbf{B}}$ which we talked about in the Introduction. Unfortunately, the author doesn't know a precise proof of this. We will be able to give other sensible definitions of $\Phi^{\mathbf{A}\to\mathbf{B}}$ in the following chapters; nonetheless, we will obtain some results of "lifting of natural transformations to homotopies": even if there is no formal justification (yet), the reader can relate them to the corresponding properties of $\Phi^{\mathbf{A}\to\mathbf{B}}$.

Dg-categories concentrated in degree 0

Let $\mathbf{A}$ be a dg-category. We say that $\mathbf{A}$ is *concentrated in degree* 0 if, for any $A, B \in \mathbf{A}$, we have $\mathbf{A}(A,B)^i = 0$ for all $i \neq 0$. We can view a given $\mathbf{k}$-linear category $\mathbf{B}$ as a dg-category concentrated in degree 0, if we endow any hom-set of $\mathbf{B}$ with the trivial graded decomposition and differential. Therefore, we obtain a fully faithful functor

$$\mathbf{k}\text{-}\mathsf{Cat} \hookrightarrow \mathsf{dgCat}, \tag{1.3.7}$$

which will be often thought as an inclusion. With this convention, if $\mathbf{A}$ is a dg-category concentrated in degree 0, we may identify $\mathbf{A} = Z^0(\mathbf{A}) = H^0(\mathbf{A})$.

Also, it is interesting to consider dg-categories with *cohomology concentrated in degree* 0, namely, dg-categories $\mathbf{A}$ such that $H^i(\mathbf{A}(A,B)) = 0$ for all $i \neq 0$, for any $A, B \in \mathbf{A}$: in such a case, a weaker form of the above identification $\mathbf{A} = H^0(\mathbf{A})$ holds. In order to make this precise, we define a dg-category $\mathbf{A}_{\leq 0}$ as follows: $\mathrm{Ob}\, \mathbf{A}_{\leq 0} = \mathrm{Ob}\, \mathbf{A}$, and for any

$A, B \in \operatorname{Ob} \mathbf{A}$ we set

$$\begin{cases} \mathbf{A}_{\leq 0}(A,B)^n = \mathbf{A}(A,B) & \text{if } n < 0, \\ \mathbf{A}_{\leq 0}(A,B)^n = 0 & \text{if } n > 0, \\ \mathbf{A}_{\leq 0}(A,B)^0 = Z^0(\mathbf{A}(A,B)), \end{cases}$$

the differential on $\mathbf{A}_{\leq 0}(A,B)$ being induced by the one on $\mathbf{A}(A,B)$; composition maps are obtained by restriction, too. There is a natural inclusion dg-functor

$$\mathbf{A}_{\leq 0} \to \mathbf{A}, \tag{1.3.8}$$

which is the identity on objects and the inclusion map $\mathbf{A}_{\leq 0}(A,B) \hookrightarrow \mathbf{A}(A,B)$ on hom-complexes ($A, B \in \operatorname{Ob} \mathbf{A}$). We also define a natural projection dg-functor

$$\mathbf{A}_{\leq 0} \to H^0(\mathbf{A}), \tag{1.3.9}$$

which is the identity on objects and the natural projection map

$$\mathbf{A}_{\leq 0}(A,B) \to H^0(\mathbf{A}(A,B))$$

on hom-complexes (here we view $H^0(\mathbf{A})$ as a dg-category concentrated in degree 0). Notice that the definitions of $\mathbf{A}_{\leq 0}$ and the above dg-functors don't require any hypothesis on $\mathbf{A}$. Nevertheless, if $\mathbf{A}$ has cohomology concentrated in degree 0, it is immediately shown that both functors are quasi-equivalences. We sum up this result in the following:

Proposition 1.3.15. *Let $\mathbf{A}$ be a dg-category such that $H^i(\mathbf{A}(X,Y)) = 0$ for any $X, Y \in \mathbf{A}$, $i \neq 0$. Then we have natural quasi-equivalences*

$$H^0(\mathbf{A}) \leftarrow \mathbf{A}_{\leq 0} \to \mathbf{A},$$

the first functor being the natural projection (1.3.9) *and the second one being the inclusion* (1.3.8). *In particular, $\mathbf{A}$ is quasi-equivalent to $H^0(\mathbf{A})$.*

Remark 1.3.16. Let Δ^1 be the standard 1-simplex $\mathbf{k}$-category, namely, the category which is freely generated over $\mathbf{k}$ by the diagram $0 \to 1$. Namely, denoting by $\mathbf{k}\langle S \rangle$ the free $\mathbf{k}$-module over a set S:

$$\begin{aligned} \Delta^1(0,1) &= \mathbf{k}, \\ \Delta^1(1,0) &= 0, \\ \Delta^1(0,0) &= \mathbf{k}\langle 1 \rangle, \\ \Delta^1(1,1) &= \mathbf{k}\langle 1 \rangle, \end{aligned}$$

and composition is given by multiplication in $\mathbf{k}$. We view it as a dg-category concentrated in degree 0. It is well-known that the category of morphisms of a $\mathbf{k}$-category $\mathbf{C}$ can be identified with $\operatorname{Fun}(\Delta^1, \mathbf{C})$. Then, it is natural to expect something similar about the dg-category of homotopy coherent morphisms $\underline{\operatorname{Mor}}\, \mathbf{A}$ of a dg-category $\mathbf{A}$. Indeed, it can be shown that there is a quasi-equivalence

$$\mathbb{R}\underline{\operatorname{Hom}}(\Delta^1, \mathbf{A}) \overset{\text{qe}}{\approx} \underline{\operatorname{Mor}}\, \mathbf{A}. \tag{1.3.10}$$

We won't give a proof of this fact; a possible technique involves the description of $\mathbb{R}\underline{\operatorname{Hom}}(\Delta^1, \mathbf{A})$ by means of A_∞-functors, as we will explain in Chapter 5.

Chapter 2

Basic dg-category theory, II

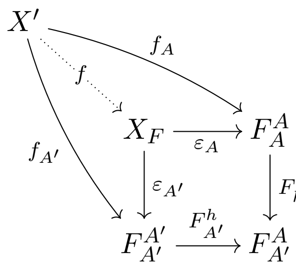
2.1 Ends and coends

Let $\mathbf{A}$ be a dg-category, and let $F\colon \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ be a dg-bi(endo)module. The aim is to construct a complex which (co)equalises the right and left actions of $\mathbf{A}$ on F. This leads to the definition of *(co)end*, given in general for dg-functors $\mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \to \mathbf{B}$. These notions will give us some very useful computational tools. This section is devoted to the development of ends and coends in dg-category theory; a good readable introduction to (co)end calculus in ordinary category theory can be found in [Lor15]. Our treatment is just a particular case of the definitions and results given in enriched category theory: possible references for the general setting are [Kel05] or [Dub70].

Definition 2.1.1. Let $F\colon \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \to \mathbf{B}$ be a dg-functor. An *end* of F is an object $X_F \in \mathbf{B}$ together with closed degree 0 maps

$$\varepsilon_A \colon X_F \to F^A_A$$

for all $A \in \mathbf{A}$, satisfying the following universal property:



(2.1.1)

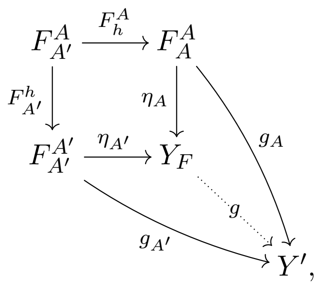
that is, for any $h \in \mathbf{A}(A, A')$ the above square with vertex X_F is commutative, and for any X' together with closed degree 0 maps $f_A\colon X' \to F^A_A$ such that the "curved square" with vertex X' is commutative, there exists a unique closed degree 0 map $f\colon X' \to X_F$ such that $f_A = \varepsilon_A f$ for all $A \in \mathbf{A}$.

Dualising, we get the definition of coend:

Definition 2.1.2. Let $F\colon \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \to \mathbf{B}$ be a dg-functor. A *coend* of F is an object $Y_F \in \mathbf{B}$ together with closed degree 0 maps

$$\eta_A\colon F_A^A \to Y_F$$

for all $A \in \mathbf{A}$, satisfying the following universal property:



(2.1.2)

that is, for any $h \in \mathbf{A}(A', A)$ the above square with vertex Y_F is commutative, and for any Y' together with closed degree 0 maps $g_A\colon F_A^A \to Y'$ such that the "curved square" with vertex Y' is commutative, there exists a unique closed degree 0 map $g\colon Y_F \to Y'$ such that $g_A = g\eta_A$ for all $A \in \mathbf{A}$.

Remark 2.1.3. Ends and coends are defined as couples $(X_F, (\varepsilon_A))$ or $(Y_F, (\eta_A))$; we will often abuse notation and refer to them as their underlying objects X_F and Y_F.

As for any object defined with a universal property, ends and coends, if they exist, are uniquely determined up to canonical isomorphism, so that we may speak of *the* (co)end of a dg-functor $F\colon \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \to \mathbf{B}$. We will adopt the integral notation: the end of F will be denoted by

$$\int_A F(A, A) = \int_A F_A^A, \tag{2.1.3}$$

and the coend of F will be denoted by

$$\int^A F(A, A) = \int^A F_A^A. \tag{2.1.4}$$

The existence of ends and coends is not assured in general; however, it holds true for bimodules, i.e. dg-functors $\mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$.

Proposition 2.1.4. *Let $F\colon \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ be an $\mathbf{A}$-$\mathbf{A}$-bimodule. Then, the end of F is isomorphic to the subcomplex of $\prod_{A \in \mathbf{A}} F(A, A)$ defined by*

$$V_F = \{\varphi = (\varphi_A)_{A \in \mathbf{A}} : f\varphi_A = (-1)^{|f||\varphi|}\varphi_{A'} f \quad \forall\, f \in \mathbf{A}(A, A')\}.$$

The map $\varepsilon_A\colon V_F \to F(A, A)$ is defined by $\varphi \mapsto \varphi_A$.

Proof. Let X be a complex, and let $\xi_A \colon X \to F(A,A)$ be chain maps such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\xi_A} & F(A,A) \\ {\scriptstyle \xi_{A'}}\downarrow & & \downarrow{\scriptstyle F(1_A \otimes f)} \\ F(A',A') & \xrightarrow{F(f\otimes 1_{A'})} & F(A,A'), \end{array}$$

for all $f \colon A \to A'$. This means that, for all $x \in X$,

$$f\xi_A(x) = (-1)^{|x||f|}\xi_{A'}(x)f,$$

employing the left-right action notation. We look for a chain map $\xi \colon X \to V_F$ such that

$$\xi(x)_A = \xi_A(x)$$

for all $x \in X$. Clearly, this identity uniquely defines ξ. Since the ξ_A are chain maps, so is ξ; the above commutativity of the ξ_A with the left and right actions ensure that ξ takes values in V_F, and we are done. □

From this characterisation we immediately obtain the following important corollary:

Corollary 2.1.5. *Let* $F, G \colon \mathbf{A} \to \mathbf{B}$ *be dg-functors. Then, the complex of dg-natural transformation* $\operatorname{Nat}_{\mathrm{dg}}(F,G)$*, together with the canonical maps*

$$\begin{aligned} \operatorname{Nat}_{\mathrm{dg}}(F,G) &\to \mathbf{B}(F(A),G(A)), \\ \varphi &\mapsto \varphi_A, \end{aligned}$$

is an end of the bimodule $h^F_G = \mathbf{B}(F(-),G(-))$*:*

$$\int_A \mathbf{B}(F(A),G(A)) \cong \operatorname{Nat}_{\mathrm{dg}}(F,G). \tag{2.1.5}$$

Proposition 2.1.6. *Let* $F \colon \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ *be an* $\mathbf{A}$*-*$\mathbf{A}$*-bimodule. Then, the coend of* F *is isomorphic to the complex*

$$W_F = \operatorname{coker}\left(\bigoplus_{A_1,A_2\in\mathbf{A}} \mathbf{A}(A_2,A_1)\otimes F(A_1,A_2) \longrightarrow \bigoplus_{A\in\mathbf{A}} F(A,A)\right)$$
$$f\otimes x \mapsto fx - (-1)^{|f||x|}xf,$$

together with the natural maps

$$\eta_A \colon F(A,A) \to \bigoplus_{A'} F(A',A') \to W_F.$$

Proof. Let X be a complex, and let $\xi_A\colon F(A,A)\to X$ be chain maps such that the following diagram is commutative:

$$\begin{array}{ccc} F(A,A') & \xrightarrow{F_f^A} & F(A,A) \\ \big\downarrow{\scriptstyle F_{A'}^f} & & \big\downarrow{\scriptstyle \xi_A} \\ F(A',A') & \xrightarrow{\xi_{A'}} & X, \end{array}$$

for all $f\colon A'\to A$. This means that, for any $x\in F(A,A')$:

$$\xi_A(fx) = (-1)^{|f||x|}\xi_{A'}(xf).$$

Hence, we find that the composition

$$\bigoplus_{A_1,A_2}\mathbf{A}(A_2,A_1)\otimes F(A_1,A_2)\to\bigoplus_A F(A,A)\xrightarrow{\oplus\xi_A} X$$

is zero, and so $\oplus\xi_A$ factors through a unique map $\xi\colon W_F\to X$. This is precisely the required universal property. □

In the following, we explore the properties of ends and coends, and we develop the tools of (co)end calculus.

Definition 2.1.7. Let $F\colon\mathbf{A}^{\mathrm{op}}\otimes\mathbf{A}\to\mathbf{B}$ and $G\colon\mathbf{B}\to\mathbf{C}$ be dg-functors. Assume that $(\int_A F(A,A),(\varepsilon_A))$ is an end of F. We say that G *preserves the end* $\int_A F(A,A)$ if $(G(\int_A F(A,A)),(G(\varepsilon_A)))$ is an end of GF. Dualising, we directly get the definition of *preservation of coends.*

We will often allow ourselves to abuse notation and write for instance

$$G(\int_A F(A,A))\cong\int_A GF(A,A),$$

to mean that G preserves the end of F. The following is an important result:

Proposition 2.1.8. *The hom-functor preserves ends. That is, given a dg-functor* $F\colon\mathbf{A}^{\mathrm{op}}\otimes\mathbf{A}\to\mathbf{B}$ *and assuming that* $\int_A F(A,A)$ *and* $\int^A F(A,A)$ *both exist, then:*

$$\mathbf{B}\left(B,\int_A F(A,A)\right)\cong\int_A\mathbf{B}(B,F(A,A)),\tag{2.1.6}$$

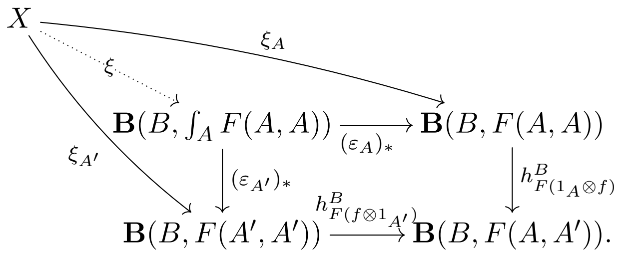
$$\mathbf{B}\left(\int^A F(A,A),B\right)\cong\int_A\mathbf{B}(F(A,A),B),\tag{2.1.7}$$

for all $B\in\mathbf{B}$.

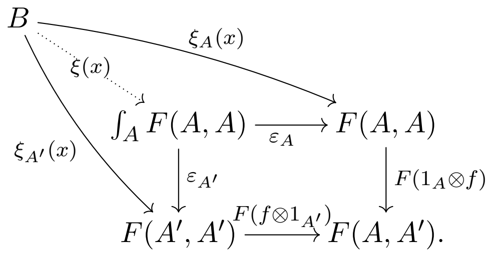
Proof. We prove (2.1.6), the other statement being dual. We have to check that, if $\varepsilon_A\colon \int_A F(A,A) \to F(A,A)$ are the canonical maps associated to the end of F, then the family of maps

$$h^B_{\varepsilon_A} = \mathbf{B}(B,\varepsilon_A) = (\varepsilon_A)_*\colon \mathbf{B}\left(B, \int_A F(A,A)\right) \to \mathbf{B}(B, F(A,A))$$

satisfies the universal property of $\int_A \mathbf{B}(B, F(A,A))$:



Let $x \in X$. Then, we have the following diagram:



So, we find a unique $\xi(x) \in Z^0(\mathbf{B}(B, \int_A F(A,A))$ such that $\varepsilon_A \circ \xi(x) = \xi_A(x)$. By this uniqueness property, we easily prove that $x \mapsto \xi(x)$ defines a chain map $X \to \mathbf{B}(B, \int_A F(A,A))$. By construction, it satisfies $(\varepsilon_A)_* \circ \xi = \xi_A$, and it is the unique with this property. $\square$

Remark 2.1.9. The isomorphisms (2.1.6) and (2.1.7) are actually a stronger (yet equivalent) version of the universal properties which define ends and coends. Recalling the characterisation of ends of $\mathsf{C}_{\mathrm{dg}}(\mathbf{k})$-valued dg-functors of Proposition 2.1.4, we see for instance that (2.1.6) is equivalent to the following statement: for any family of maps $\xi_A\colon B \to F(A,A)$ of degree p such that the following diagram

$$\begin{array}{ccc} B & \xrightarrow{\xi_A} & F(A,A) \\ {\scriptstyle \xi_{A'}}\downarrow & & \downarrow{\scriptstyle F(1_A\otimes f)} \\ F(A',A') & \xrightarrow{F(f\otimes 1_{A'})} & F(A,A') \end{array}$$

is commutative up to the sign $(-1)^{pq}$, for all $f\colon A \to A'$ of degree q, there exists a unique $\xi\colon B \to \int_A F(A,A)$ of degree p such that $\varepsilon_A \xi = \xi_A$ for all $A \in \mathbf{A}$.

We are now able to show that ends and coends are dg-functorial. We write down statements only for ends, the case of coends being analogous.

Proposition 2.1.10 (Dg-functoriality)**.** *Let $F, G \colon \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \to \mathbf{B}$ be dg-functors, and assume that $\int_A F(A,A)$ and $\int_A G(A,A)$ exist. If $\varphi \colon F \to G$ is a dg-natural transformation, then there exists a natural morphism*

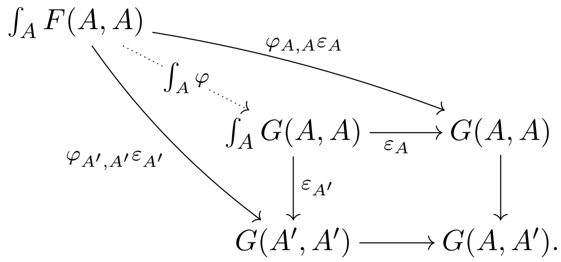
$$\int_A \varphi \colon \int_A F(A,A) \to \int_A G(A,A).$$

The mapping $\varphi \mapsto \int_A \varphi$ is a chain map, and moreover we have

$$\int_A \psi\varphi = \int_A \psi \circ \int_A \varphi,$$
$$\int_A 1_F = 1_{\int_A F(A,A)},$$

assuming $\psi \colon G \to H$ and the existence of $\int_A H(A,A)$.

Proof. Define $\int_A \varphi$ with the strong universal property explained in Remark 2.1.9:



The properties required follow by uniqueness arguments. □

Remark 2.1.11. Let $F \colon \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \otimes \mathbf{C} \to \mathbf{B}$ be a dg-functor. Assume that, for all $C \in \mathbf{C}$, the end $\int_A F(A,A,C)$ exists. Then, it is dg-functorial in C. indeed, given $f \colon C \to C'$, we obtain a dg-natural transformation

$$\varphi_f = F(-,-,f) \colon F(-,-,C) \to F(-,-,C')$$

of functors $\mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \to \mathbf{B}$, and by dg-functoriality we get a natural morphism:

$$\int_A \varphi_f \colon \int_A F(A,A,C) \to \int_A F(A,A,C').$$

The mapping $f \mapsto \varphi_f \mapsto \int_A \varphi_f$ is dg-functorial, so in the end we get a dg-functor

$$\int_A F(A,A,-) \colon \mathbf{C} \to \mathbf{B},$$

together with natural transformations

$$\varepsilon_A \colon \int_A F(A,A,-) \to F(A,A,-).$$

This is the "end with parameters". The same discussion can obviously be done for coends.

The following result is an "interchange law" for (co)ends. With the integral notation, it becomes a "categorical Fubini theorem". We give the statement for ends:

Proposition 2.1.12 ("Fubini theorem")**.** *Let $F\colon \mathbf{A}^{\mathrm{op}} \otimes \mathbf{B}^{\mathrm{op}} \otimes \mathbf{A} \otimes \mathbf{B} \to \mathbf{C}$ be a dg-functor. Assume that for all $A, A' \in \mathbf{A}$, the end*

$$\int_B F^{A,B}_{A',B}$$

exists. Then, there is a natural isomorphism:

$$\int_{(A,B)} F^{A,B}_{A,B} \cong \int_A \int_B F^{A,B}_{A,B},$$

whenever one of these two ends exists. Moreover, if for all $B, B' \in \mathbf{B}$, the end $\int_A F^{A,B}_{A,B'}$ also exists, then

$$\int_{(A,B)} F^{A,B}_{A,B} \cong \int_A \int_B F^{A,B}_{A,B} \cong \int_B \int_A F^{A,B}_{A,B}, \tag{2.1.8}$$

whenever one of these ends exist.

Proof. Assume that $\int_{(A,B)} F^{A,B}_{A,B}$ exists, and let $\varepsilon_{(A,B)}\colon \int_{(A,B)} F^{A,B}_{A,B} \to F^{A,B}_{A,B}$ be the natural associated maps. Apply the universal property of $\int_B F^{A,B}_{A,B}$ (together with the associated maps $p_B\colon \int_B F^{A,B}_{A,B} \to F^{A,B}_{A,B}$):

$$\begin{array}{ccc} \int_{(A,B)} F^{A,B}_{A,B} & \xrightarrow{\varepsilon_{(A,B)}} & F^{A,B}_{A,B} \\ {\scriptstyle q_A}\downarrow & \nearrow{\scriptstyle p_B} & \\ \int_B F^{A,B}_{A,B}. & & \end{array}$$

So, $(\int_{A,B} F^{A,B}_{A,B}, (q_A))$ satisfies the universal property of $\int_A \int_B F^{A,B}_{A,B}$. Conversely, assume that $(\int_A \int_B F^{A,B}_{A,B}, (q_A))$ is an end of $\int_B F^{-,B}_{-,B}$; then, define $\varepsilon_{(A,B)}\colon \int_A \int_B F^{A,B}_{A,B} \to F^{A,B}_{A,B}$ as $\varepsilon_{(A,B)} = p_B q_A$, and check that they satisfy the universal property of $\int_{(A,B)} F^{A,B}_{A,B}$. We leave the reader to fill in the details, and to conclude with the proof of (2.1.8). $\square$

Yoneda lemma, revisited

The complex of natural transformations between two dg-functors can be written as an end, as we have already seen in Corollary 2.1.5. So, it is clear that Yoneda lemma can be restated employing this formalism. This also allows us to give a proof which is completely based on the suitable universal property.

Proposition 2.1.13 (Yoneda lemma)**.** *Let $\mathbf{A}$ be a dg-category, let $F\colon \mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ and $G\colon \mathbf{A}^{\mathrm{op}} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ be respectively a left and a right $\mathbf{A}$-dg-module. Then:*

$$\begin{aligned} F_- &\cong \int_A \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(h^-_A, F_A), \\ G^- &\cong \int_A \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(h^A_-, G^A). \end{aligned} \tag{2.1.9}$$

where the natural maps $\varepsilon_A\colon F_- \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(h^-_A, F_A)$ *and* $\varepsilon'_A\colon G^- \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(h^A_-, G^A)$ *are defined respectively by*

$$\begin{aligned} \varepsilon_A(x)(f) &= (-1)^{|x||f|} fx, \\ \varepsilon'_A(y)(g) &= yg. \end{aligned}$$

Proof. We prove the second statement, the other one being dual. Let V be a complex, let $X \in \mathbf{A}$ and let $\xi_A\colon V \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(h^A_X, G^A)$ be chain maps such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\xi_A} & \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(h^A_X, G^A) \\ \downarrow{\scriptstyle \xi_{A'}} & & \downarrow{\scriptstyle (G^g)_*} \\ \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(h^{A'}_X, G^{A'}) & \xrightarrow{(h^g_X)^*} & \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(h^A_X, G^{A'}) \end{array}$$

is commutative for all $g\colon A' \to A$. We want to define $\xi\colon V \to G^X$ such that $\varepsilon'_A \xi = \xi_A$ for all $A \in \mathbf{A}$. In particular, we require that

$$\varepsilon'_A(\xi(v))(h) = \xi(v)h = (-1)^{|v||h|} G(h)(\xi(v)) = \xi_A(v)(h),$$

for all $h\colon A \to X$. So, we are forced to set

$$\xi(v) = \xi_X(v)(1_X).$$

We leave the reader to check that this definition satisfies the required properties (it is a rather tedious but straightforward verification).

Assuming we have already proved Yoneda lemma with the usual direct argument, we can give a shorter proof of the isomorphisms (2.1.9): indeed, again concentrating on the second one, we just have to check that the diagram

$$\begin{array}{ccc} \mathrm{Nat}_{\mathrm{dg}}(h_X, G) & \xrightarrow{\sim} & G^X \\ \downarrow & \swarrow{\scriptstyle \varepsilon'_A} & \\ \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(h^A_X, G^A) & & \end{array} \qquad \begin{array}{ccc} \varphi & \longmapsto & \varphi_X(1_X) \\ \downarrow & \swarrow & \\ \varphi_A & & \end{array}$$

is commutative. $\square$

Interestingly, there is a dual version of Yoneda lemma, which involves coends:

Proposition 2.1.14 (Co-Yoneda lemma)**.** *Let* $\mathbf{A}$ *be a dg-category, let* $F\colon \mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ *and* $G\colon \mathbf{A}^{\mathrm{op}} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ *be respectively a left and a right* $\mathbf{A}$*-dg-module. Then:*

$$\begin{aligned} F_- &\cong \int^A h^A_- \otimes F_A, \\ G^- &\cong \int^A G^A \otimes h^-_A, \end{aligned} \tag{2.1.10}$$

where the associated maps $\eta_A \colon h^A_- \otimes F_A \to F_-$ *and* $\eta'_A \colon G^A \otimes h^-_A \to G^-$ *are induced by the (left and right) actions of* $\mathbf{A}$*:*

$$\eta_A(f \otimes x) = fx,$$
$$\eta'_A(y \otimes g) = yg.$$

Proof. We prove only the first isomorphism, the other one being dual. Let V be a complex, and let $X \in \mathbf{A}$. We have the following chain of natural isomorphisms:

$$\begin{aligned}
&\int_A \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(h^A_X \otimes F_A, V) \\
&\qquad \cong \int_A \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(h^A_X, \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(F_A, V)) \\
&\qquad \overset{\text{(Yon.)}}{\cong} \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(F_X, V).
\end{aligned}$$

This implies that F_X represents the dg-functor $V \mapsto \int_A \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(h^A_X \otimes F_A, V)$, and so by definition (recall the "strong universal property, Remark 2.1.9) it is the expected coend:

$$F_X \cong \int^A h^A_X \otimes F_A.$$

To understand what are the associated maps η_A, we follow the above chain backwards, starting from the unit 1_{F_X} and keeping track of its image:

$$\begin{aligned}
1_{F_X} &\mapsto (\varepsilon_A(1_X)(f) = F(f)^*(1_X) = F(f)) \\
&\mapsto (\eta_A(f \otimes x) = F(f)(x) = fx).
\end{aligned}$$
□

The above proof follows a typical pattern in (co)end calculus. To show that a certain object X is a (co)end, we try to prove that it represents the suitable functor, and in doing so we make use of the computational tools developed so far: (co)end preservation, dg-functoriality, Fubini theorem, and so on. Typically, we end up writing a chain of natural isomorphisms. At every step, we should keep track of the natural maps associated to the written (co)end; the isomorphisms of the chain will always preserve them, and this knowledge allows us to understand what are the natural maps associated to the object X, as we did in the second part of the above proof.

2.2 (Co)complete dg-categories

Just as in ordinary category theory, there is a notion of completeness and cocompleteness also in dg-category theory, which stems from the definitions given in general enriched category theory. Our treatment here will be based on (co)ends and (co)tensors. Ends and coends have already been extensively studied in the previous section; tensors and cotensors are "external versions" respectively of the tensor product of complexes and of the internal hom in the monoidal category $\mathsf{C}(\mathbf{k})$.

Definition 2.2.1. Let $\mathbf{A}$ be a dg-category, and let $V \in \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ and $A \in \mathbf{A}$. A *tensor* (or *copower*) of V and A is an object $V \otimes A \in \mathbf{A}$ together with an isomorphism of complexes:

$$\mathbf{A}(V \otimes A, B) \xrightarrow{\sim} \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(V, \mathbf{A}(A, B)), \tag{2.2.1}$$

natural in B. Dually, a *cotensor* (or *power*) of V and A is an object $[V, A] \in \mathbf{A}$ together with an isomorphism of complexes:

$$\mathbf{A}(B, [V, A]) \xrightarrow{\sim} \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(V, \mathbf{A}(B, A)), \tag{2.2.2}$$

natural in B.

If $\mathbf{A}$ contains all tensors, we say that it is *tensored* (or *copowered*). Dually, if it contains all cotensors, we say that it is *cotensored* (or *powered*).

Remark 2.2.2. The dg-category $\mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ is tensored and cotensored. Tensors $V \otimes W$ are given by the ordinary tensor products, whereas cotensors $[V, W]$ are given by the internal hom-complexes $\mathsf{C}_{\mathrm{dg}}(\mathbf{k})(V, W)$.

If $\mathbf{A}$ is tensored, then the above definition yields a unique dg-bifunctor

$$- \otimes - \colon \mathsf{C}_{\mathrm{dg}}(\mathbf{k}) \otimes \mathbf{A} \to \mathbf{A} \tag{2.2.3}$$

such that the isomorphism (2.2.1) becomes natural in all variables. Dually, if $\mathbf{A}$ is cotensored, then we get a unique dg-bifunctor

$$[-, -] \colon \mathsf{C}_{\mathrm{dg}}(\mathbf{k})^{\mathrm{op}} \otimes \mathbf{A} \to \mathbf{A} \tag{2.2.4}$$

such that the isomorphism (2.2.2) becomes natural in all variables. Moreover, we have dg-adjunctions:

$$- \otimes A \dashv \mathbf{A}(A, -) \colon \mathsf{C}_{\mathrm{dg}}(\mathbf{k}) \leftrightarrows \mathbf{A}, \tag{2.2.5}$$

$$\mathbf{A}(-, A) \dashv [-, A] \colon \mathbf{A}^{\mathrm{op}} \leftrightarrows \mathsf{C}_{\mathrm{dg}}(\mathbf{k}), \tag{2.2.6}$$

for all $A \in \mathbf{A}$, if $\mathbf{A}$ is respectively tensored or cotensored.

Now, we are able to define (co)completeness for dg-categories:

Definition 2.2.3. Let $\mathbf{A}$ be a dg-category. We say that $\mathbf{A}$ is *complete* if it is cotensored and any dg-functor $F \colon \mathbf{B}^{\mathrm{op}} \otimes \mathbf{B} \to \mathbf{A}$ (with $\mathbf{B}$ small) admits an end $\int_B F(B, B)$ in $\mathbf{A}$.

Dually, we say that $\mathbf{A}$ is *cocomplete* if it is tensored and any dg-functor $F \colon \mathbf{B}^{\mathrm{op}} \otimes \mathbf{B} \to \mathbf{A}$ (with $\mathbf{B}$ small) admits a coend $\int^B F(B, B)$ in $\mathbf{A}$.

The dg-category of complexes $\mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ is complete and cocomplete, since it is tensored and cotensored, and we have already proven that ends and coends of bimodules $\mathbf{B}^{\mathrm{op}} \otimes \mathbf{B} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ always exists. The (co)completeness of a given dg-category implies the (co)completeness of the dg-category of functors with values in that dg-category:

Proposition 2.2.4. *Let* $\mathbf{B}$ *and* $\mathbf{A}$ *be dg-categories. If* $\mathbf{A}$ *is complete (resp. cocomplete) then so is* $\mathrm{Fun}_{\mathrm{dg}}(\mathbf{B}, \mathbf{A})$. *In particular, the dg-category* $\mathsf{C}_{\mathrm{dg}}(\mathbf{B})$ *is both complete and cocomplete.*

Proof. The idea is to define (co)ends and (co)tensors componentwise. Let $F\colon \mathbf{C}^{\mathrm{op}} \otimes \mathbf{C} \to \mathrm{Fun}_{\mathrm{dg}}(\mathbf{B}, \mathbf{A})$. We may view it as a functor $F\colon \mathbf{C}^{\mathrm{op}} \otimes \mathbf{C} \otimes \mathbf{B} \to \mathbf{A}$. So, by Remark 2.1.11, the (co)end of F exists if and only if the (co)end of $F(-,-,B)$ exists for all $B \in \mathbf{B}$. Coming to (co)tensors, it should be clear that they are obtained from (co)tensors in $\mathbf{A}$, componentwise:

$$\begin{aligned}(V \otimes F)(B) &= V \otimes F(B),\\ [V, F](B) &= [V, F(B)],\end{aligned}$$

whenever $F \in \mathrm{Fun}_{\mathrm{dg}}(\mathbf{B}, \mathbf{A})$ and $V \in \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$. □

Having defined (co)completeness as above, we are now interested in functors which preserve (co)tensors and (co)ends. The notion of (co)end preservation was given in Definition 2.1.7; (co)tensor preservation is defined in the obvious way, as follows. Given a dg-functor $F\colon \mathbf{A} \to \mathbf{B}$, we say that F preserves the tensor $V \otimes A$ if $F(V \otimes A)$ is naturally a tensor of V and $F(A)$ in $\mathbf{B}$: $F(V \otimes A) \cong V \otimes F(A)$; dually, we say that F preserves the cotensor $[V, A]$ if $F([V, A])$ is naturally a cotensor of V and $F(A)$: $F([V, A]) \cong [V, F(A)]$.

Definition 2.2.5. Let $F\colon \mathbf{A} \to \mathbf{B}$ be a dg-functor. We say that F is *continuous* if it preserves all ends and cotensors. Dually, we say that F is *cocontinuous* if it preserves all coends and tensors.

Remark 2.2.6. (Co)completeness for enriched categories (in particular, dg-categories) – and (co)continuity of enriched functors (in particular, dg-functors) – is usually given in terms of the existence – or preservation – of *weighted (co)limits*: they are the reasonable replacement of (co)limits in enriched category theory. Those weighted (co)limits can be expressed in terms of (co)ends and (co)tensors (see [Rie14, Theorem 7.6.3]) and vice-versa (see [Lor15, Example 4.11]). A detailed treatment of enriched (co)completeness is contained in [Rie14, Sections 7.4 and 7.6] and [Kel05, Section 3.2].

The universal properties (2.2.2) and (2.2.1) directly tell us that the hom functors $\mathbf{A}(-, A)$ and $\mathbf{A}(A, -)$ are continuous, since we have already proven in Proposition 2.1.8 end and coend preservation (beware that $\mathbf{A}(-, A)$ is contravariant, so it actually maps tensors to cotensors and coends to ends). As in ordinary category theory, adjoint functors have (co)continuity properties:

Proposition 2.2.7. *Let $F \dashv G\colon \mathbf{A} \leftrightarrows \mathbf{B}$ be adjoint dg-functors. Then, F is cocontinuous and G is continuous.*

Proof. It is a direct computation involving the continuity of the hom functor, precisely as in ordinary category theory. The details are left to the reader. □

Remark 2.2.8. The tensor product of complexes, being left adjoint to the internal hom (and being symmetric), is cocontinuous in both variables. This observation will be useful later on.

Extensions of dg-modules

Here we recollect the basic results about restrictions and extensions of dg-modules along dg-functors. Actually, they are a particular case of the more general notion of *Kan extensions.*

Definition 2.2.9. Let $K\colon \mathbf{A} \to \mathbf{B}$ and $F\colon \mathbf{A} \to \mathbf{C}$ be dg-functors. A *left Kan extension* of F along K is a dg-functor $\operatorname{Lan}_K(F)\colon \mathbf{B} \to \mathbf{C}$ together with an isomorphism:

$$\operatorname{Nat}_{\mathrm{dg}}(\operatorname{Lan}_K(F), G) \xrightarrow{\sim} \operatorname{Nat}_{\mathrm{dg}}(F, G \circ K),$$

natural in $G\colon \mathbf{B} \to \mathbf{C}$. Dually, a *right Kan extension* of F along K is a dg-functor $\operatorname{Ran}_K(F)\colon \mathbf{B} \to \mathbf{C}$ together with an isomorphism:

$$\operatorname{Nat}_{\mathrm{dg}}(G \circ K, F) \xrightarrow{\sim} \operatorname{Nat}_{\mathrm{dg}}(G, \operatorname{Ran}_K(F)),$$

natural in $G\colon \mathbf{B} \to \mathbf{C}$.

Clearly, left and right Kan extensions are characterised by universal properties. For example, $\operatorname{Lan}_K(F)$ comes with a natural transformation $\eta\colon F \to \operatorname{Lan}_K(F) \circ K$ such that for any $\alpha\colon F \to G \circ K$ there exists a unique $\widetilde{\alpha}\colon \operatorname{Lan}_K(F) \to G$ such that $\alpha = (\widetilde{\alpha}K)\eta$. So, in short, Kan extensions are "lax universal extensions" along a given dg-functor. If $\operatorname{Lan}_K(F)$ and $\operatorname{Ran}_K(F)$ exist for all $F\colon \mathbf{A} \to \mathbf{C}$, then $\operatorname{Lan}_K(-)$ and $\operatorname{Ran}_K(-)$ define left and right dg-adjoints to the restriction functor $K^* = \operatorname{Res}_K\colon \operatorname{Fun}_{\mathrm{dg}}(\mathbf{B}, \mathbf{C}) \to \operatorname{Fun}_{\mathrm{dg}}(\mathbf{A}, \mathbf{C})$:

$$\operatorname{Lan}_K \dashv K^* \dashv \operatorname{Ran}_K . \tag{2.2.7}$$

(Co)completeness properties of the target dg-category $\mathbf{C}$ give sufficient conditions to the existence of Kan extensions. More precisely:

Proposition 2.2.10. *Assume that* $\mathbf{C}$ *is cocomplete. Then,* $\operatorname{Lan}_K(F)$ *exists for all dg-functors* $F\colon \mathbf{A} \to \mathbf{C}$*, and it is given by*

$$\operatorname{Lan}_K(F)(-) \cong \int^A h_-^{K(A)} \otimes F_A. \tag{2.2.8}$$

Dually, assume that $\mathbf{C}$ *is complete. Then,* $\operatorname{Ran}_K(F)$ *exists for all* $F\colon \mathbf{A} \to \mathbf{C}$*, and it is given by*

$$\operatorname{Ran}_K(F)(-) \cong \int_A [h^-_{K(A)}, F_A]. \tag{2.2.9}$$

Proof. We only prove the assertion about Ran, the other one being similar. Let $G\colon \mathbf{B} \to$

$\mathbf{C}$ be a dg-functor. We compute directly:

$$\begin{aligned}
\mathrm{Nat}_{\mathrm{dg}}(G, \mathrm{Ran}_K(F)) &\cong \int_B \mathbf{C}(G_B, \mathrm{Ran}_K(F)(B)) \\
&\cong \int_B \mathbf{C}(G_B, \int_A [h^B_{K(A)}, F_A]) \\
&\cong \int_B \int_A \mathbf{C}(G_B, [h^B_{K(A)}, F_A]) \\
&\cong \int_A \int_B \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(h^B_{K(A)}, \mathbf{C}(G_B, F_A)) \\
&\cong \int_A \mathbf{C}(G_{K(A)}, F_A) \\
&\cong \mathrm{Nat}_{\mathrm{dg}}(G \circ K, F).
\end{aligned}$$

Every isomorphism of the above chain is natural; we used end calculus and the definition of the power $[h^B_{K(A)}, F_A]$. □

From now on, for the sake of simplicity, we will always assume that the target dg-category $\mathbf{C}$ is complete and cocomplete, so that Kan extensions always exist. the typical situations are $\mathbf{C} = \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ and $\mathbf{C} = \mathrm{Fun}_{\mathrm{dg}}(\mathbf{D}, \mathsf{C}_{\mathrm{dg}}(\mathbf{k}))$ (dg-categories of dg-modules).

If the dg-functor $K\colon \mathbf{A} \to \mathbf{B}$ along which we extend is fully faithful, then Kan extensions are actual (not just lax) extensions:

Proposition 2.2.11. *Let $K\colon \mathbf{A} \to \mathbf{B}$ be a fully faithful dg-functor. Then, for any $F\colon \mathbf{A} \to \mathbf{C}$, the units*

$$\begin{aligned}
\eta_F &\colon F \to \mathrm{Lan}_K(F) \circ K, \\
\varepsilon_F &\colon \mathrm{Ran}_K(F) \circ K \to F
\end{aligned}$$

are natural isomorphisms. In particular, the dg-functors Lan_K and Ran_K are fully faithful.

Proof. By Proposition 1.2.14, it suffices to show that there are isomorphisms $F \cong \mathrm{Lan}_K(F) \circ K$ and $F \cong \mathrm{Ran}_K(F) \circ K$, natural in F. We detail the first one, the other being similar. Let $G\colon \mathbf{A} \to \mathbf{C}$. We have:

$$\begin{aligned}
\mathrm{Nat}_{\mathrm{dg}}(\mathrm{Lan}_K(F)(K(-)), G) &\cong \int_{A'} \mathbf{C}\left(\int^A h^{K(A)}_{K(A')} \otimes F_A, G_{A'}\right) \\
&\cong \int_A \int_{A'} \mathbf{C}(h^{K(A)}_{K(A')} \otimes F_A, G_{A'}) \\
&\cong \int_A \int_{A'} \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(h^{K(A)}_{K(A')}, \mathbf{C}(F_A, G_{A'})) \\
&\cong \int_A \int_{A'} \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(h^A_{A'}, \mathbf{C}(F_A, G_{A'})) \\
&\cong \int_A \mathbf{C}(F_A, G_A) \\
&\cong \mathrm{Nat}_{\mathrm{dg}}(F, G).
\end{aligned}$$

□

In the case when $\mathbf{C} = \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$, the above discussion gives the general results about extensions of dg-modules. We write

$$\mathrm{Ind}_F \colon \mathsf{C}_{\mathrm{dg}}(\mathbf{A}) \to \mathsf{C}_{\mathrm{dg}}(\mathbf{B}) \tag{2.2.10}$$

for the left Kan extension $\mathrm{Lan}_{F^{\mathrm{op}}}$ along $F^{\mathrm{op}} \colon \mathbf{A}^{\mathrm{op}} \to \mathbf{B}^{\mathrm{op}}$. It is the left adjoint of the restriction functor $\mathrm{Res}_F \colon \mathsf{C}_{\mathrm{dg}}(\mathbf{B}) \to \mathsf{C}_{\mathrm{dg}}(\mathbf{A})$. By Proposition 2.2.10, we may write (swapping the arguments of the tensor):

$$\mathrm{Ind}_F(M)^- \cong \int^A M^A \otimes h^-_{F(A)}. \tag{2.2.11}$$

The extension functor Ind_F preserves representable $\mathbf{A}$-modules:

Proposition 2.2.12. *The diagram*

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{F} & \mathbf{B} \\ \big\downarrow{\scriptstyle h_{\mathbf{A}}} & & \big\downarrow{\scriptstyle h_{\mathbf{B}}} \\ \mathsf{C}_{\mathrm{dg}}(\mathbf{A}) & \xrightarrow{\mathrm{Ind}_F} & \mathsf{C}_{\mathrm{dg}}(\mathbf{B}) \end{array}$$

is commutative up to isomorphism. In other words, $\mathrm{Ind}_F(h_A) \cong h_{F(A)}$ *for all* $A \in \mathbf{A}$*.*

Proof. We compute:

$$\begin{aligned} &\mathrm{Ind}_F(h_A)^X \\ &\quad \cong \int^{A'} h_A^{A'} \otimes h^X_{F(A')} \\ &\quad \cong h^X_{F(A)}, \end{aligned}$$

where the last isomorphism follows from co-Yoneda lemma (2.1.10). □

2.3 Pretriangulated dg-categories

The relevance of dg-categories relies in the crucial observation that they can be employed as higher categorical models for triangulated categories. The key point is that we have a notion of functorial shifts of objects, and – above all – of functorial cones of (closed, degree 0) morphisms.

Definition 2.3.1. Let $\mathbf{A}$ be a dg-category, $A \in \mathbf{A}$ and $n \in \mathbb{Z}$. A *n-shift of A* is an object $A[n] \in \mathbf{A}$ together with two closed morphisms

$$1_{(A,n,0)} \colon A[n] \to A, \quad 1_{(A,0,n)} \colon A \to A[n], \tag{2.3.1}$$

such that $|1_{(A,n,0)}| = n$, $|1_{(A,0,n)}| = -n$ and

$$1_{(A,n,0)} 1_{(A,0,n)} = 1_A, \quad 1_{(A,0,n)} 1_{(A,n,0)} = 1_{A[n]}.$$

Given a morphism f in $\mathbf{A}$, we will often employ the notation

$$f[n] = 1_{(B,0,n)} f 1_{(A,n,0)} \colon A[n] \to B[n] \quad (n \in \mathbb{Z}). \tag{2.3.2}$$

A n-shift, if it exists, is unique up to a natural dg-isomorphism: if $A[n]$ and $A[n]'$ are two n-shifts of an object $A \in \mathbf{A}$, with associated morphisms $1_{(A,n,0)}, 1_{(A,0,n)}$ and $1'_{(A,n,0)}, 1'_{(A,0,n)}$, then clearly $1'_{(A,0,n)} 1_{(A,n,0)}$ and $1_{(A,0,n)} 1'_{(A,n,0)}$ are closed degree 0 maps, each one inverse to the other.

Next, we define the *cone* of a degree 0 morphism: it is a direct generalization of the mapping cone of a chain map of complexes.

Definition 2.3.2. Let $\mathbf{A}$ be a dg-category and let $f \colon A \to B$ be a closed degree 0 morphism in $\mathbf{A}$; suppose moreover that $\mathbf{A}$ contains the shift $A[1]$. A *cone* of f is an object $\mathrm{C}(f) \in \mathbf{A}$ together with degree 0 morphisms

$$A[1] \underset{p}{\overset{i}{\rightleftarrows}} \mathrm{C}(f) \underset{j}{\overset{s}{\rightleftarrows}} B, \tag{2.3.3}$$

such that

$$pi = 1_{A[1]}, \quad sj = 1_B, \quad si = 0, \quad pj = 0, \quad ip + js = 1_{\mathrm{C}(f)}, \tag{2.3.4}$$

and

$$dj = 0, \quad dp = 0, \quad di = jf1_{(A,1,0)}, \quad ds = -f1_{(A,1,0)}p. \tag{2.3.5}$$

Remark 2.3.3. Conditions (2.3.4) tell us that any cone $\mathrm{C}(f)$ of f is a biproduct of $A[1]$ and B in $\mathbf{A}$, viewed as a k-linear category, with canonical maps i, j, p, s. We shall adopt matrix notation when working with maps to or from a cone:

$$u = (u_1, u_2) \colon \mathrm{C}(f) \to D$$

means that $u \circ i = u_1$ and $u \circ j = u_2$, and

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : D \to \mathrm{C}(f)$$

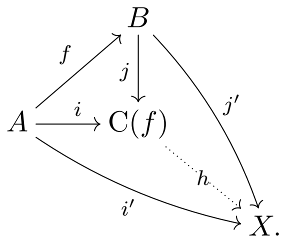
means that $p \circ v = v_1$ and $s \circ v = v_2$. Conditions (2.3.5) allow us to easily describe the differential of maps to and from a cone. Namely, if $u = (u_1, u_2) \colon \mathrm{C}(f) \to D$ and $v = \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right) : D \to \mathrm{C}(f)$, for a given $f \colon A \to B$, then

$$du = (du_1 - (-1)^{|u|} u_2 f 1_{(A,1,0)}, du_2), \quad dv = \begin{pmatrix} dv_1 \\ dv_2 + f1_{(A,1,0)} v_1 \end{pmatrix}. \tag{2.3.6}$$

The cone of a morphism satisfies a universal property, as follows:

Proposition 2.3.4. *Let* $\mathbf{A}$ *be a dg-category, and let* $f \colon A \to B$ *be a closed degree* 0 *morphism. View the inclusion map* $i \colon A[1] \to \mathrm{C}(f)$ *as a degree* -1 *morphism* $A \to \mathrm{C}(f)$. *Then,* $di = jf$, *and for any closed degree* 0 *map* $j' \colon B \to X$ *and any degree* -1 *map*

$i' \colon A \to X$ such that $di' = j'f$, there exists a unique closed degree 0 map $h \colon \mathrm{C}(f) \to X$ such that $hj = j'$ and $hi = i'$:



(2.3.7)

Proof. Since $\mathrm{C}(f)$ is a biproduct of $A[1]$ and B in $\mathbf{A}$ viewed as an ordinary category, we are forced to define:

$$h = (i' 1_{(A,1,0)}, j').$$

The degree of h is clearly 0. Then, (2.3.6) allows us to compute the differential of h, and show immediately that $dh = 0$. □

The above universal property, as every universal property, implies that the cone of a morphism is uniquely determined up to a unique dg-isomorphism. We remark that we could also define $\mathrm{C}(f)$ by means of the above universal property, and obtain our definition as a characterisation.

Remark 2.3.5. It is worth noticing that shifts and cones can be characterised by means of tensors and coends. Let $S^n = \mathbf{k}[n]$ be the complex with the only term $\mathbf{k}$ concentrated in degree $-n$. Moreover, let D^n be the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{k} = \mathbf{k} \longrightarrow 0 \longrightarrow \cdots,$$

where the first $\mathbf{k}$ is in degree $-n$. There is a natural inclusion $S^n \hookrightarrow D^{n+1}$:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbf{k} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & \downarrow & & \| & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathbf{k} & = & \mathbf{k} & \longrightarrow & 0 & \longrightarrow & \cdots . \end{array}$$

We remark that we have a natural isomorphism of complexes:

$$\mathsf{C}_{\mathrm{dg}}(\mathbf{k})(S^n, V) \cong V[-n], \quad f \mapsto f(1).$$

Let $\mathbf{A}$ be a dg-category, and let $A \in \mathbf{A}$. The tensor $S^n \otimes A$ can be identified with the shift $A[n]$, indeed:

$$\begin{aligned} \mathbf{A}(S^n \otimes A, B) &\cong \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(S^n, \mathbf{A}(A, B)) \\ &\cong \mathbf{A}(A, B)[-n] \\ &\cong \mathbf{A}(A[n], B), \end{aligned}$$

where the last isomorphism is induced by composition with $1_{(A,n,0)}$. Moreover, the tensor $D^{n+1} \otimes A$ can be identified with the cone $\mathrm{C}(1_{A[n]})$ of the identity on $A[n]$ (it is a rather simple exercise: just check the universal property by comparing the formulae of differentials (2.3.6) and (1.1.3)).

Now, let $f\colon A \to B$ be a closed degree 0 morphism in $\mathbf{A}$. We identify it to a dg-functor $f\colon \Delta^1 \to \mathbf{A}$ such that $f(0) = A$ and $f(1) = B$. Moreover, let $W\colon (\Delta^1)^{\mathrm{op}} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ the dg-functor defined as follows:

$$W(0) = D^1, \quad W(1) = S^0, \quad W((0 \to 1)^{\mathrm{op}}) = S^0 \hookrightarrow D^1.$$

Then, we have that

$$\mathrm{C}(f) \cong \int^{i \in \Delta^1} W(i) \otimes f(i), \tag{2.3.8}$$

assuming that $\mathbf{A}$ is tensored. In order to prove this, we start by noticing that the diagram

$$\begin{array}{ccc} W(1) \otimes f(0) & \longrightarrow & W(1) \otimes f(1) \\ \big\downarrow & & \\ W(0) \otimes f(0) & & \end{array}$$

translates to

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \big\downarrow{\scriptstyle j_A} & & \\ \mathrm{C}(1_A), & & \end{array}$$

where $j_A = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ is the canonical inclusion of A in $\mathrm{C}(1_A)$. Now, we have closed and degree 0 morphisms $j\colon B \to \mathrm{C}(f)$ and $i_1\colon \mathrm{C}(1_A) \to \mathrm{C}(f)$. The first one is the canonical inclusion map into $\mathrm{C}(f)$, whereas the second one is defined as follows:

$$i_1 = (i, jf)\colon \mathrm{C}(1_A) \to \mathrm{C}(f),$$

where $i\colon A[1] \to \mathrm{C}(f)$ is the canonical inclusion map. Since $di = jf1_{(A,1,0)}$, the above map is closed, and moreover the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \big\downarrow{\scriptstyle \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} & & \big\downarrow{\scriptstyle j} \\ \mathrm{C}(1_A) & \xrightarrow{i_1} & \mathrm{C}(f) \end{array}$$

is commutative. Also, notice that any degree 0 map $v = (v_1, v_2)\colon \mathrm{C}(1_A) \to X$ is closed if and only if $v_2 1_{(A,1,0)} = dv_1$, and any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \big\downarrow{\scriptstyle \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} & & \big\downarrow{\scriptstyle u} \\ \mathrm{C}(1_A) & \xrightarrow{v} & X \end{array}$$

gives $dv_1 = uf1_{(A,1,0)}$. Now, it should be clear that the universal property of the coend (2.3.8) is the same as the universal property (2.3.7) of $\mathrm{C}(f)$.

A sequence of the form

$$A \xrightarrow{f} B \xrightarrow{j} \mathrm{C}(f) \xrightarrow{p} A[1] \tag{2.3.9}$$

is called *preexact triangle*, or *pretriangle*; notice that any dg-functor preserves shifts, cones and preexact triangles. Now, we can give the following definition:

Definition 2.3.6. Let $\mathbf{A}$ be a dg-category. We say that $\mathbf{A}$ is *strongly pretriangulated* if any object $A \in \mathbf{A}$ admits all shifts $A[n]$ in $\mathbf{A}$, and if $\mathbf{A}$ contains the cone of any closed degree 0 morphism.

The above Remark 2.3.5 immediately gives the following:

Proposition 2.3.7. *Cocomplete dg-categories are strongly pretriangulated. In particular, $\mathsf{C}_{\mathrm{dg}}(\mathbf{k})$ is strongly pretriangulated, and $\mathsf{C}_{\mathrm{dg}}(\mathbf{A})$ is strongly pretriangulated for any dg-category $\mathbf{A}$.*

Also, an "objectwise" argument similar to that of Proposition 2.2.4 easily yields the following:

Proposition 2.3.8. *Let $\mathbf{A}$ and $\mathbf{B}$ be dg-categories. If $\mathbf{B}$ is strongly pretriangulated, then $\mathrm{Fun}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})$ is strongly pretriangulated.*

Pretriangulated hulls

A feature of the definition of strongly pretriangulated dg-category is that it is "intrinsic", in the sense that it is not an additional structure we endow something with, as happens with triangulated categories. Moreover, the rigidity of dg-categories allows us to make useful functorial constructions. A simple way to define them employs the Yoneda embedding $\mathbf{A} \hookrightarrow \mathsf{C}_{\mathrm{dg}}(\mathbf{A})$, as in the following definition:

Definition 2.3.9. Let $\mathbf{A}$ be a dg-category. The *pretriangulated hull* or *pretriangulated envelope* of $\mathbf{A}$ is the dg-category $\mathrm{pretr}(\mathbf{A})$, which is by definition the smallest full dg-subcategory of $\mathsf{C}_{\mathrm{dg}}(\mathbf{A})$ which contains the image of $\mathbf{A}$ under the Yoneda embedding, and which is strongly pretriangulated and closed under dg-isomorphisms.

In other words, $\mathrm{pretr}(\mathbf{A})$ is obtained from $\mathbf{A}$ by adding shifts and cones of morphisms between representable modules. The Yoneda embedding can be restricted to a dg-functor

$$\mathbf{A} \hookrightarrow \mathrm{pretr}(\mathbf{A}). \tag{2.3.10}$$

We have the following characterisation:

Proposition 2.3.10. *A dg-category is strongly pretriangulated if and only if the restricted Yoneda embedding $\mathbf{A} \hookrightarrow \mathrm{pretr}(\mathbf{A})$ is a dg-equivalence.*

Proof. Clearly, if $\mathbf{A} \hookrightarrow \operatorname{pretr}(\mathbf{A})$ is a dg-equivalence, then $\mathbf{A}$ is strongly pretriangulated, being dg-equivalent to a strongly pretriangulated dg-category. Conversely, assume that $\mathbf{A}$ is strongly pretriangulated. Then, its closure under dg-isomorphisms in $\mathsf{C}_{\mathrm{dg}}(\mathbf{A})$ is equivalent to $\mathbf{A}$, and it coincides with $\operatorname{pretr}(\mathbf{A})$, by definition of pretriangulated hull. □

As we may expect, a dg-functor $F\colon \mathbf{A} \to \mathbf{B}$ induces a dg-functor $\operatorname{pretr}(\mathbf{A}) \to \operatorname{pretr}(\mathbf{B})$ between the pretriangulated hulls. Indeed, consider the induction dg-functor $\operatorname{Ind}_F\colon \mathsf{C}_{\mathrm{dg}}(\mathbf{A}) \to \mathsf{C}_{\mathrm{dg}}(\mathbf{B})$. Then, since any $X \in \operatorname{pretr}(\mathbf{A})$ is obtained from representable $\mathbf{A}$-modules by means of iterated shifts and cones and Ind_F preserves shifts and cones (being a dg-functor), we see that $\operatorname{Ind}_F(X) \in \operatorname{pretr}(\mathbf{B})$, so in the end it induces a dg-functor

$$\operatorname{pretr}(F)\colon \operatorname{pretr}(\mathbf{A}) \to \operatorname{pretr}(\mathbf{B}). \tag{2.3.11}$$

Moreover, by Proposition 2.2.12, the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{F} & \mathbf{B} \\ \downarrow{\scriptstyle \theta_{\mathbf{A}}} & & \downarrow{\scriptstyle \theta_{\mathbf{B}}} \\ \operatorname{pretr}(\mathbf{A}) & \xrightarrow{\operatorname{pretr}(F)} & \operatorname{pretr}(\mathbf{B}), \end{array} \tag{2.3.12}$$

where $\theta_{\mathbf{A}}$ and $\theta_{\mathbf{B}}$ are the restricted Yoneda embeddings of $\mathbf{A}$ and $\mathbf{B}$.

Now, we can prove that the pretriangulated hull of a dg-category satisfies a 2-categorical universal property, which characterises it up to dg-equivalence:

Proposition 2.3.11. *Let* $\mathbf{A}$ *be a dg-category, and let* $\mathbf{B}$ *be a strongly pretriangulated dg-category. Then, the restricted Yoneda embedding* $\theta_{\mathbf{A}}\colon \mathbf{A} \hookrightarrow \operatorname{pretr}(\mathbf{A})$ *induces a natural dg-equivalence:*

$$\theta_{\mathbf{A}}^*\colon \operatorname{Fun}_{\mathrm{dg}}(\operatorname{pretr}(\mathbf{A}), \mathbf{B}) \xrightarrow{\sim} \operatorname{Fun}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B}). \tag{2.3.13}$$

Proof. We give just the main ideas. A natural transformation of dg-functors $F \to G\colon \operatorname{pretr}(\mathbf{A}) \to \mathbf{B}$ is uniquely determined by its restriction $F\theta_{\mathbf{A}} \to G\theta_{\mathbf{A}}\colon \mathbf{A} \to \mathbf{B}$: this is because $\operatorname{pretr}(\mathbf{A})$ is obtained from $\mathbf{A}$ by iterated shifts and cones, which are functorial, and the dg-functors F and G preserve them. This proves fully faithfulness of $\theta_{\mathbf{A}}^*$. Essential surjectivity is simple: given $F\colon \mathbf{A} \to \mathbf{B}$, then by (2.3.12) we immediately see that $\theta_{\mathbf{B}}^{-1} \circ \operatorname{pretr}(F)$ is in the essential inverse image of F. It should be noticed that a quasi-inverse of $\theta_{\mathbf{A}}^*$ gives a left Kan extension of F along $\theta_{\mathbf{A}}$. □

Another important feature of the pretriangulated hull is that it behaves well under quasi-equivalences:

Lemma 2.3.12 ([BLL04, Remark 4.12])**.** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be dg-categories. If* $F\colon \mathbf{A} \to \mathbf{B}$ *is a quasi-equivalence, then* $\operatorname{pretr}(F)\colon \operatorname{pretr}(\mathbf{A}) \to \operatorname{pretr}(\mathbf{B})$ *is a quasi-equivalence.*

Until now, we have explored the "strict" notion of pretriangulated category: in a strongly pretriangulated dg-categories, we require the existence of shifts and cones up to dg-isomorphism. This is somewhat against the philosophy of homotopy theory, for which the relevant constructions should be the homotopy invariant ones. So, we want to define a dg-category to be *pretriangulated* if it contains (functorial) cones and shifts *up to homotopy equivalence*. This can be formalised as follows, recalling the characterisation of Proposition 2.3.10:

Definition 2.3.13. Let $\mathbf{A}$ be a dg-category. We say that $\mathbf{A}$ is *pretriangulated* if the restricted Yoneda embedding

$$\theta_{\mathbf{A}} \colon \mathbf{A} \hookrightarrow \operatorname{pretr}(\mathbf{A})$$

is a quasi-equivalence.

We shall come back to this notion in the following chapter; now, we just mention the fundamental result which links pretriangulated dg-categories to triangulated categories:

Theorem 2.3.14. *If $\mathbf{A}$ is pretriangulated, then $H^0(\mathbf{A})$ is a triangulated category, endowed with the triangulated structure coming from $\mathbf{A}$. Moreover, if $F \colon \mathbf{A} \to \mathbf{B}$ is a dg-functor between pretriangulated categories, $H^0(F) \colon H^0(\mathbf{A}) \to H^0(\mathbf{B})$ is an exact functor.*

In particular, the category $\mathsf{K}(\mathbf{B})$ is triangulated for any dg-category $\mathbf{B}$.

The proof of this theorem is quite long yet straightforward: it is analogous to the proof that the homotopy category of complexes $\mathsf{K}(\mathbf{k})$ is triangulated. This serves as a motivation of the following definition:

Definition 2.3.15. Let $\mathbf{T}$ be a triangulated category. A *dg-enhancement* (or simply *enhancement*) of $\mathbf{T}$ is a pretriangulated dg-category $\mathbf{A}$ together with an exact equivalence $\epsilon \colon H^0(\mathbf{A}) \to \mathbf{T}$.

The theoretical drawbacks of triangulated categories have motivated the research on enhancements (existence and uniqueness problem). This is not the main interest of this work, nonetheless we will encounter many examples of enhancement throughout the thesis. Now, we just end the section (and the chapter) with a very simple characterisation of quasi-equivalences between pretriangulated dg-categories:

Lemma 2.3.16. *Let $\mathbf{A}$ and $\mathbf{B}$ be pretriangulated dg-categories, and let $F \colon \mathbf{A} \to \mathbf{B}$ be a dg-functor. Then, F is a quasi-equivalence if and only if $H^0(F)$ is an equivalence of triangulated categories.*

Proof. Left as an exercise: actually only shifts are needed. □

Corollary 2.3.17. *Let $\mathbf{A}$ and $\mathbf{B}$ be pretriangulated dg-categories, moreover let $\widehat{F} \in \mathsf{Hqe}(\mathbf{A}, \mathbf{B})$. Then, F is an isomorphism in Hqe if and only if $H^0(\widehat{F})$ is an equivalence.*

Proof. Represent $\widehat{F}$ as a roof:

$$\mathbf{A} \xleftarrow{\approx} Q(\mathbf{A}) \xrightarrow{F'} \mathbf{B}.$$

If $\widehat{F}$ is an isomorphism, then the above F' is a quasi-equivalence, so in the end $H^0(\widehat{F})$ is an equivalence. Conversely, if $H^0(\widehat{F})$ is an equivalence, then $H^0(F')$ is such, and so by Lemma 2.3.16 it is a quasi-equivalence. We conclude that $\widehat{F}$ is an isomorphism in Hqe, being represented by a roof of quasi-equivalences. □

Chapter 3

Quasi-functors

In this chapter we go deeper into the homotopy theory of dg-categories: in particular, we address the fundamental issue of *quasi-functors*, that is, the relevant "homotopy coherent functors" between dg-categories. A way to describe them is by employing bimodules and their derived categories. This approach also allows us to give a consistent notion of adjoint quasi-functors.

3.1 The derived category

Let $F \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A})$ be a right $\mathbf{A}$-dg-module. An important operation that can be performed on F is taking its cohomology $H^*(F)$, which is defined termwise:

$$H^*(F)(A) = H^*(F(A)).$$

$H^*(F)$ is actually a right graded $H^*(\mathbf{A})$-module, that is, a graded functor

$$H^*(F) \colon H^*(\mathbf{A}^{\mathrm{op}}) \to \underline{\mathsf{Gr}}(\mathbf{k})$$

(here $\underline{\mathsf{Gr}}(\mathbf{k})$ denotes the graded category of graded $\mathbf{k}$-modules). The right action of $H^*(\mathbf{A})$ on $H^*(F)$ is defined by

$$[x][f] = [xf],$$

projecting the action of $\mathbf{A}$ to the quotient. Denoting by $\underline{\mathsf{Gr}}(H^*(\mathbf{A}))$ the graded category of graded right $H^*(\mathbf{A})$-modules, we obtain a graded functor

$$\begin{aligned} H^* \colon Z^*(\mathsf{C}_{\mathrm{dg}}(\mathbf{A})) &\to \underline{\mathsf{Gr}}(H^*(\mathbf{A})), \\ F &\mapsto H^*(F), \\ (\alpha \colon F \to G) &\mapsto H^*(\alpha), \end{aligned}$$

where $H^*(\alpha)$ is defined by $H^*(\alpha)_A([x]) = [\alpha_A(x)]$. Restricting ourselves to degree 0 morphisms, we also get an ordinary functor

$$H^* \colon \mathsf{C}(\mathbf{A}) \to \mathsf{Gr}(H^*(\mathbf{A})),$$

where $\mathsf{Gr}(H^*(\mathbf{A}))$ is the (ordinary) category of graded right $H^*(\mathbf{A})$-modules.

Remark 3.1.1. The above $H^*\colon Z^*(\mathsf{C}_{\mathrm{dg}}(\mathbf{A})) \to \underline{\mathsf{Gr}}(H^*(\mathbf{A}))$ factors through the projection functor $Z^*(\mathsf{C}_{\mathrm{dg}}(\mathbf{A})) \to H^*(\mathsf{C}_{\mathrm{dg}}(\mathbf{A}))$. Indeed, if a closed morphism $\alpha\colon F \to G$ satisfies $[\alpha] = [0]$ in $\mathsf{K}(\mathbf{A})$, then $H^*(\alpha) = 0$. We abuse notation and write

$$H^*\colon H^*(\mathsf{C}_{\mathrm{dg}}(\mathbf{A})) \to \underline{\mathsf{Gr}}(H^*(\mathbf{A}))$$

also for the induced functor. Restricting ourselves to degree 0 morphisms, we also get

$$H^*\colon \mathsf{K}(\mathbf{A}) \to \mathsf{Gr}(H^*(\mathbf{A})).$$

The above discussion can be specialised to degree 0 cohomology, yielding a $\mathbf{k}$-linear functor

$$H^0\colon \mathsf{C}(\mathbf{A}) \to \mathsf{Mod}(H^0(\mathbf{A})),$$

where the category $\mathsf{Mod}(H^0(\mathbf{A}))$ is the $\mathbf{k}$-linear category of right $H^0(\mathbf{A})$-modules (that is, $\mathbf{k}$-linear functors $H^0(\mathbf{A})^{\mathrm{op}} \to \mathsf{Mod}(\mathbf{k})$). As above, we also have the induced functor, defined on the homotopy category:

$$H^0\colon \mathsf{K}(\mathbf{A}) \to \mathsf{Mod}(H^0(\mathbf{A})). \tag{3.1.1}$$

Remark 3.1.2. We warn the reader of the potential confusion arising between projecting a closed morphism $\alpha\colon F \to G$ of right $\mathbf{A}$-modules to $[\alpha]\colon F \to G$ in $H^*(\mathsf{C}_{\mathrm{dg}}(\mathbf{A}))$, and taking its cohomology $H^*(\alpha)$, which is a (graded) morphism of graded modules $H^*(F) \to H^*(G)$. $[\alpha]$ is just the cohomology class of a "strict" morphism α, whereas $H^*(\alpha)$ is the actual graded cohomology of α, which in fact depends only on $[\alpha]$. So, $[\alpha]$ is not a natural transformation but, instead, an equivalence class of natural transformations; $H^*(\alpha)$, on the other hand, is a natural transformation at the homotopy level.

Definition 3.1.3. Let $\alpha\colon F \to G$ be a morphism in $\mathsf{C}(\mathbf{A})$. We say that α is a *quasi-isomorphism* if

$$H^*(\alpha)\colon H^*(F) \to H^*(G)$$

is an isomorphism.

Now, we would like to identify dg-modules up to quasi-isomorphism. Naively, given $\mathbf{A}$-dg-modules F, G, we define $F \sim G$ if there exists a quasi-isomorphism $\alpha\colon F \to G$. This relation is reflexive (the identity map is a quasi-isomorphism) and transitive (the composite of two quasi-isomorphisms is a quasi-isomorphism), but unfortunately it is not symmetric. So, we define the quasi-isomorphism relation as the equivalence relation generated by the above $\sim$. In particular, F and G are quasi-isomorphic if there exists a zig-zag of quasi-isomorphisms:

$$F \leftarrow F_1 \to F_2 \leftarrow \cdots \to G;$$

in this case, we write $F \overset{\mathrm{qis}}{\approx} G$. The machinery of localisations applies to this framework, and leads to the following definition:

Definition 3.1.4. Let $\mathbf{A}$ be a dg-category. The *derived category* $\mathsf{D}(\mathbf{A})$ of $\mathbf{A}$ is the localisation of $\mathsf{K}(\mathbf{A})$ along quasi-isomorphisms:

$$\mathsf{D}(\mathbf{A}) = \mathsf{K}(\mathbf{A})[\mathrm{Qis}^{-1}]. \tag{3.1.2}$$

When $\mathbf{A} = \mathbf{k}$, viewing the base ring as a dg-category with a single object, its derived category is by definition the derived category $\mathsf{D}(\mathbf{k})$ of complexes of $\mathbf{k}$-modules. The well-known results about $\mathsf{D}(\mathbf{k})$ have a direct generalisation to $\mathsf{D}(\mathbf{A})$ for any $\mathbf{A}$. We recollect them in the following statement.

Definition 3.1.5. Let $\mathbf{A}$ be a dg-category. A right $\mathbf{A}$-dg-module $M \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A})$ is *acyclic* if $M(A)$ is an acyclic complex for all $A \in \mathbf{A}$ (or, equivalently, if $H^*(M) \cong 0$). The full dg-subcategory of $\mathsf{C}_{\mathrm{dg}}(\mathbf{A})$ of acyclic modules is denoted by $\mathrm{Ac}(\mathbf{A})$.

Proposition 3.1.6 ([Kel06, Lemma 3.3])**.** *$\mathsf{D}(\mathbf{A})$ has a natural structure of triangulated category such that the localisation functor $\delta = \delta_{\mathbf{A}} \colon \mathsf{K}(\mathbf{A}) \to \mathsf{D}(\mathbf{A})$ is exact.*

A morphism $\alpha \colon F \to G$ in $\mathsf{K}(\mathbf{A})$ is a quasi-isomorphism if and only if its cone $\mathrm{C}(\alpha)$ is acyclic. Moreover, $\mathsf{D}(\mathbf{A})$ is the Verdier quotient of $\mathsf{K}(\mathbf{A})$ modulo the acyclic modules:

$$\mathsf{D}(\mathbf{A}) \cong \mathsf{K}(\mathbf{A})/\mathrm{Ac}(\mathbf{A}). \tag{3.1.3}$$

The localisation functor $\delta \colon \mathsf{K}(\mathbf{A}) \to \mathsf{D}(\mathbf{A})$ can (and will) be assumed to be the identity on objects. Morphisms in $\mathsf{D}(\mathbf{A})$ are represented by "roofs" in $\mathsf{K}(\mathbf{A})$:

$$F \xleftarrow{\approx} F' \to G, \tag{3.1.4}$$

where $F' \xrightarrow{\approx} F$ is a quasi-isomorphism. The idea is that F' is suitable *resolution* of F. In particular, we may assume it is a *h-projective resolution*:

Definition 3.1.7. Let $F \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A})$. F is *h-projective* if, for all $N \in \mathrm{Ac}(\mathbf{A})$, the complex $\mathsf{C}_{\mathrm{dg}}(\mathbf{A})(F, N)$ is acyclic. This equivalent to requiring that

$$H^0(\mathsf{C}_{\mathrm{dg}}(\mathbf{A}))(F, N) = \mathsf{K}(\mathbf{A})(F, N) \cong 0$$

for all $N \in \mathrm{Ac}(\mathbf{A})$. The full dg-subcategory of $\mathsf{C}_{\mathrm{dg}}(\mathbf{A})$ of h-projective dg-modules is denoted by $\text{h-proj}(\mathbf{A})$.

H-projective modules can be characterised as follows:

Proposition 3.1.8 ([BL94, Proposition 10.12.2.2])**.** *Let $P \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A})$ be a right $\mathbf{A}$-dg-module. Then, $P \in \text{h-proj}(\mathbf{A})$ if and only if*

$$\delta \colon \mathsf{K}(\mathbf{A})(P, M) \to \mathsf{D}(\mathbf{A})(P, M)$$

is an isomorphism for all $M \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A})$.

Remark 3.1.9. From the above proposition, we see that any quasi-isomorphism between h-projective dg-modules is actually a homotopy equivalence.

The following result ensures the existence of h-projective resolutions, and explains their features:

Proposition 3.1.10 ([Kel06, Proposition 3.1])**.** *Any dg-module F admits a* h-projective resolution*, that is, a quasi-isomorphism*

$$q_F\colon Q(F) \xrightarrow{\approx} F, \tag{3.1.5}$$

natural in $F \in \mathsf{K}(\mathbf{A})$, where $Q(F)$ is h-projective. Moreover, Q yields a fully faithful left adjoint $Q\colon \mathsf{D}(\mathbf{A}) \to \mathsf{K}(\mathbf{A})$ to the localisation functor $\delta\colon \mathsf{K}(\mathbf{A}) \to \mathsf{D}(\mathbf{A})$. The adjunction is obtained as follows:

$$\mathsf{K}(\mathbf{A})(Q(M), N) \xrightarrow{\delta} \mathsf{D}(\mathbf{A})(Q(M), N) \xrightarrow{(\delta(q_M)^{-1})^*} \mathsf{D}(\mathbf{A})(M, N). \tag{3.1.6}$$

Lemma 3.1.11. h-proj$(\mathbf{A})$ *is a strongly pretriangulated full dg-subcategory of the dg-category* $\mathsf{C}_{\mathrm{dg}}(\mathbf{A})$*.*

Proof. It is sufficient to show that, if $P \in$ h-proj$(\mathbf{A})$, then $P[i] \in$ h-proj$(\mathbf{A})$ for all $i \in \mathbb{Z}$, and that $\mathrm{C}(f) \in$ h-proj$(\mathbf{A})$ whenever $f\colon P \to P'$ is a closed degree 0 morphism between h-projective modules. Let $M \in \mathrm{Ac}(\mathbf{A})$. Then:

$$\mathsf{K}(\mathbf{A})(P[i], M) \cong \mathsf{K}(\mathbf{A})(P, M[-i]) \cong 0,$$

because $M[-i] \in \mathrm{Ac}(\mathbf{A})$, so $P[i] \in$ h-proj$(\mathbf{A})$. Moreover, since $\mathsf{K}(\mathbf{A})$ is a triangulated category, there is an exact sequence:

$$\cdots \to \mathsf{K}(\mathbf{A})(P[1], M) \to \mathsf{K}(\mathbf{A})(\mathrm{C}(f), M) \to \mathsf{K}(\mathbf{A})(P', M) \to \mathsf{K}(\mathbf{A})(P, M) \to \cdots.$$

So, since $\mathsf{K}(\mathbf{A})(P, M)$ and $\mathsf{K}(\mathbf{A})(P', M)$ are zero, the same is true for $\mathsf{K}(\mathbf{A})(\mathrm{C}(f), M)$. This tells us that $\mathrm{C}(f) \in$ h-proj$(\mathbf{A})$. □

Corollary 3.1.12. h-proj$(\mathbf{A})$ *is an enhancement of the derived category* $\mathsf{D}(\mathbf{A})$*. More precisely, the functor*

$$H^0(\text{h-proj}(\mathbf{A})) \hookrightarrow \mathsf{K}(\mathbf{A}) \xrightarrow{\delta} \mathsf{D}(\mathbf{A}) \tag{3.1.7}$$

is an equivalence.

The above discussion can be dualised. In fact, morphisms $F \to G$ in $\mathsf{D}(\mathbf{A})$ can also be represented as "coroofs":

$$F \xrightarrow{\approx} R(F) \leftarrow G,$$

where $F \to R(F)$ is a *h-injective resolution*. The results discussed above have their obvious counterparts. For the reader's convenience, we state the definition of h-injective dg-module and the analogue of Proposition 3.1.10:

Definition 3.1.13. Let $\mathbf{A}$ be a dg-category, and let $F \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A})$. F is *h-injective* if, for all $N \in \mathrm{Ac}(\mathbf{A})$, the complex $\mathsf{C}_{\mathrm{dg}}(\mathbf{A})(N, F)$ is acyclic. This equivalent to requiring that

$$H^0(\mathsf{C}_{\mathrm{dg}}(\mathbf{A}))(N, F) = \mathsf{K}(\mathbf{A})(N, F) \cong 0$$

for all $N \in \mathrm{Ac}(\mathbf{A})$. The full dg-subcategory of $\mathsf{C}_{\mathrm{dg}}(\mathbf{A})$ of h-injective dg-modules is denoted by h-inj$(\mathbf{A})$.

Proposition 3.1.14 ([Kel06, Proposition 3.1])**.** *Every dg-module F admits a* h-injective resolution*, that is, a quasi-isomorphism*

$$r_F \colon F \xrightarrow{\approx} R(F), \tag{3.1.8}$$

natural in $F \in \mathsf{K}(\mathbf{A})$, where $R(F)$ is h-injective. Moreover, Q yields a fully faithful right adjoint $R \colon \mathsf{D}(\mathbf{A}) \to \mathsf{K}(\mathbf{A})$ to the localisation functor $\delta \colon \mathsf{K}(\mathbf{A}) \to \mathsf{D}(\mathbf{A})$. The adjunction is obtained as follows:

$$\mathsf{D}(\mathbf{A})(M,N) \xrightarrow{\delta(r_N)_*} \mathsf{D}(\mathbf{A})(M,R(N)) \xrightarrow{\delta^{-1}} \mathsf{K}(\mathbf{A})(M,R(N)). \tag{3.1.9}$$

Remark 3.1.15. If M is an h-projective dg-module, then we may assume without loss of generality that $Q(M) = M$. Analogously, if N is an h-injective dg-module, we may assume that $R(M) = M$.

Moreover, notice that h-projectives (and their resolutions) can be defined also in the opposite category $\mathsf{K}(\mathbf{A})^{\mathrm{op}}$: they coincide with h-injectives (and their resolutions) in $\mathsf{K}(\mathbf{A})$, and vice-versa.

Derived functors and derived adjunctions

Let $\mathbf{A}$ and $\mathbf{B}$ be dg-categories, and let $F \colon \mathsf{K}(\mathbf{A}) \to \mathsf{K}(\mathbf{B})$ be an exact functor (in most situation, it is induced by a dg-functor). A typical question is the following: does F induce an exact functor $F' \colon \mathsf{D}(\mathbf{A}) \to \mathsf{D}(\mathbf{B})$ such that the diagram

$$\begin{array}{ccc} \mathsf{K}(\mathbf{A}) & \xrightarrow{F} & \mathsf{K}(\mathbf{B}) \\ \downarrow{\scriptstyle \delta_{\mathbf{A}}} & & \downarrow{\scriptstyle \delta_{\mathbf{B}}} \\ \mathsf{D}(\mathbf{A}) & \xrightarrow{F'} & \mathsf{D}(\mathbf{B}) \end{array}$$

is commutative? The answer is positive if F preserves acyclic $\mathbf{A}$-modules (or, equivalently, quasi-isomorphisms). In this case, the induced functor F' is often identified with F itself.

In many situations, however, our given functor $F \colon \mathsf{K}(\mathbf{A}) \to \mathsf{K}(\mathbf{B})$ does not preserve acyclics; nevertheless, it always does when restricted to h-projective (or h-injective) dg-modules:

Lemma 3.1.16. *Let $F \colon \mathsf{K}(\mathbf{A}) \to \mathsf{K}(\mathbf{B})$ be an exact functor. Then, F maps dg-modules which are both acyclic and h-projective (resp. acyclic and h-injective) to acyclics, or equivalently it preserves quasi-isomorphisms between h-projective (resp. h-injective) dg-modules.*

Proof. Cones of morphisms between h-projective or h-injective dg-modules are easily seen to be themselves h-projective or h-injective. So, recalling that quasi-isomorphisms are precisely the morphisms whose cone is acyclic, it is clear that F preserves quasi-isomorphisms between h-projectives (resp. h-injectives) if and only if it maps dg-modules which are both acyclic and h-projective (resp. acyclic and h-injective) to acyclics. Now, let $M \in \mathsf{K}(\mathbf{A})$ be h-projective and acyclic (or h-injective and acyclic). By h-projectivity (or h-injectivity) any quasi-isomorphism $M \to 0$ is actually a homotopy equivalence. So, $F(M) \approx F(0) = 0$ in $\mathsf{K}(\mathbf{B})$, in particular it is acyclic. □

Now, even if our functor $F\colon \mathsf{K}(\mathbf{A}) \to \mathsf{K}(\mathbf{B})$ does not pass to the derived categories, it induces the so-called *derived functors*. Abstractly, they are defined as Kan extensions[1]:

Definition 3.1.17. Let $F\colon \mathsf{K}(\mathbf{A}) \to \mathsf{K}(\mathbf{B})$ be a functor. A *(total) left derived functor* $\mathbb{L}F$ of F is a *right* Kan extension of $\delta_{\mathbf{B}} \circ F$ along $\delta_{\mathbf{A}}$:

$$\begin{array}{ccc} \mathsf{K}(\mathbf{A}) & \xrightarrow{F} & \mathsf{K}(\mathbf{B}) \\ \downarrow{\scriptstyle \delta_{\mathbf{A}}} & \Uparrow & \downarrow{\scriptstyle \delta_{\mathbf{B}}} \\ \mathsf{D}(\mathbf{A}) & \xrightarrow[\mathbb{L}F]{} & \mathsf{D}(\mathbf{B}). \end{array}$$

Dually, a *(total) right derived functor* $\mathbb{R}F$ of F is a *left* Kan extension of $\delta_B \circ F$ along δ_A.

Clearly derived functors, being Kan extensions, are uniquely determined up to isomorphism. The above Lemma 3.1.16 ensures that derived functors actually exist in our framework, and the following proposition gives their concrete definitions, which is what we will actually use. Its proof can be found in [Rie14, Theorem 2.2.8], in a more general setting.

Proposition 3.1.18. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be dg-categories, and let* $F\colon \mathsf{K}(\mathbf{A}) \to \mathsf{K}(\mathbf{B})$ *be an exact functor. We know that* F *preserves quasi-isomorphisms between h-projectives; then,* F *admits a left derived functor, obtained as follows:*

$$\begin{gathered} \mathbb{L}F\colon \mathsf{D}(\mathbf{A}) \to \mathsf{D}(\mathbf{B}), \\ \mathbb{L}F = \delta_{\mathbf{B}} \circ F \circ Q_{\mathbf{A}} \end{gathered} \tag{3.1.10}$$

where $Q_{\mathbf{A}}\colon \mathsf{D}(\mathbf{A}) \to \mathsf{K}(\mathbf{A})$ *is the h-projective resolution functor of* $\mathbf{A}$.

Dually, we know that F *preserves quasi-isomorphisms between h-injectives; then,* F *admits a right derived functor, obtained as follows:*

$$\begin{gathered} \mathbb{R}F\colon \mathsf{D}(\mathbf{A}) \to \mathsf{D}(\mathbf{B}), \\ \mathbb{R}F = \delta_{\mathbf{B}} \circ F \circ R_{\mathbf{A}}, \end{gathered} \tag{3.1.11}$$

where $R_{\mathbf{A}}\colon \mathsf{D}(\mathbf{A}) \to \mathsf{K}(\mathbf{A})$ *is the h-injective resolution functor of* $\mathbf{A}$.

Remark 3.1.19. We have observed that, if the functor $F\colon \mathsf{K}(\mathbf{A}) \to \mathsf{K}(\mathbf{B})$ preserves acyclics, then it directly induces a functor $F\colon \mathsf{D}(\mathbf{A}) \to \mathsf{D}(\mathbf{B})$ between the derived categories. In this case, we don't need to derive F, indeed we immediately see that

$$F \cong \mathbb{L}F \cong \mathbb{R}F\colon \mathsf{D}(\mathbf{A}) \to \mathsf{D}(\mathbf{B}).$$

[1]Here, we mean Kan extensions of ordinary ($\mathbf{k}$-linear) functors: they are defined in a formally identical way as Kan extensions of dg-functors.

We will often encounter adjunctions between categories of dg-modules. As expected, just as functors can be derived, the same is true for adjunctions:

Proposition 3.1.20. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be dg-categories, and let*

$$F \dashv G \colon \mathsf{K}(\mathbf{A}) \leftrightarrows \mathsf{K}(\mathbf{B})$$

be an adjunction of exact functors. Then, there is a derived adjunction

$$\mathbb{L}F \dashv \mathbb{R}G \colon \mathsf{D}(\mathbf{A}) \leftrightarrows \mathsf{D}(\mathbf{B}), \tag{3.1.12}$$

which is obtained composing the three adjunctions $Q_{\mathbf{A}} \dashv \delta_A$*,* $F \dashv G$ *and* $\delta_{\mathbf{B}} \dashv R_{\mathbf{B}}$*. Namely:*

$$\begin{aligned} \mathsf{D}(\mathbf{B})(\mathbb{L}F(M), N) &= \mathsf{D}(\mathbf{B})(\delta_{\mathbf{B}} F Q_{\mathbf{A}}(M), N) \\ &\cong \mathsf{K}(\mathbf{B})(F Q_{\mathbf{A}}(M), R_{\mathbf{B}}(N)) \\ &\cong \mathsf{K}(\mathbf{A})(Q_{\mathbf{A}}(M), G R_{\mathbf{B}}(N)) \\ &\cong \mathsf{D}(\mathbf{A})(M, \delta_{\mathbf{A}} G R_{\mathbf{B}}(N)) \\ &= \mathsf{D}(\mathbf{A})(M, \mathbb{R}G(N)). \end{aligned} \tag{3.1.13}$$

We conclude the discussion showing a typical example of derived adjunction. Let $F \colon \mathbf{A} \to \mathbf{B}$. We know that the restriction functor $\operatorname{Res}_F \colon \mathsf{C}_{\mathrm{dg}}(\mathbf{B}) \to \mathsf{C}_{\mathrm{dg}}(\mathbf{A})$ has a left adjoint:

$$\operatorname{Ind}_F \dashv \operatorname{Res}_F \colon \mathsf{C}_{\mathrm{dg}}(\mathbf{A}) \leftrightarrows \mathsf{C}_{\mathrm{dg}}(\mathbf{B}).$$

If $M \in \mathsf{C}_{\mathrm{dg}}(\mathbf{B})$ is acyclic, then its restriction $\operatorname{Res}_F(M)$ is acyclic. So, Res_F induces an exact functor between the derived categories, which we also call Res_F:

$$\operatorname{Res}_F \colon \mathsf{D}(\mathbf{B}) \to \mathsf{D}(\mathbf{A}).$$

The functor Ind_F can be derived, yielding

$$\mathbb{L}\operatorname{Ind}_F \colon \mathsf{D}(\mathbf{A}) \to \mathsf{D}(\mathbf{B}). \tag{3.1.14}$$

The restriction functor doesn't need to be derived, so we directly obtain the adjunction

$$\mathbb{L}\operatorname{Ind}_F \dashv \operatorname{Res}_F .$$

As a final remark, we point out that $\mathbb{L}\operatorname{Ind}_F$ preserves representable modules, just as Ind_F (Proposition 2.2.12). In fact, since any representable h_A is h-projective (we will prove this in the beginning of the next section), we may assume $Q(h_A) = h_A$, so that in the end we have:

$$\mathbb{L}\operatorname{Ind}_F(h_A) \overset{\mathrm{qis}}{\approx} h_{F(A)}.$$

3.2 The derived Yoneda embedding

Let $\mathbf{A}$ be a dg-category, and let $A \in \mathbf{A}$. Then, the right $\mathbf{A}$-module h_A is h-projective. Indeed, let $N \in \mathrm{Ac}(\mathbf{A})$ be an acyclic module. Then, by Yoneda lemma:

$$\mathsf{K}(\mathbf{A})(h_A, N) \cong H^0(N^A) \cong 0.$$

So, we see that the Yoneda embedding $h_{\mathbf{A}}$ factors through h-proj($\mathbf{A}$), yielding

$$h_{\mathbf{A}} \colon \mathbf{A} \to \text{h-proj}(\mathbf{A}). \tag{3.2.1}$$

Taking H^0 and composing with the equivalence $H^0(\text{h-proj}(\mathbf{A})) \xrightarrow{\sim} \mathsf{D}(\mathbf{A})$ of Corollary 3.1.12, we obtain the so-called *derived Yoneda embedding*:

$$H^0(\mathbf{A}) \hookrightarrow \mathsf{D}(\mathbf{A}). \tag{3.2.2}$$

By definition, the essential image of this functor is the category qrep($\mathbf{A}$) of quasi-representable right $\mathbf{A}$-modules. Moreover, the derived Yoneda embedding allows us to define “triangulated envelopes” of $\mathbf{A}$. The following definition follows Keller’s terminology (see [Kel06]):

Definition 3.2.1. Let $\mathbf{A}$ be a dg-category. The *triangulated category associated to* $\mathbf{A}$ is the smallest strictly full[2] triangulated subcategory of $\mathsf{D}(\mathbf{A})$ which contains the essential image of (3.2.2), and it is denoted by tria($\mathbf{A}$). Moreover, the *category of perfect objects* per($\mathbf{A}$) is the smallest strictly full thick[3] triangulated subcategory of $\mathsf{D}(\mathbf{A})$ which contains the essential image of (3.2.2).

Remark 3.2.2. The triangulated category per($\mathbf{A}$) is *idempotent complete*, that is, every map $e \colon X \to X$ such that $e^2 = e$ splits, namely, $e = gf$ for suitable morphisms f and g such that $fg = 1$. Moreover, it can be shown that it is the *idempotent completion* of tria($\mathbf{A}$).

Both tria($\mathbf{A}$) and per($\mathbf{A}$) have dg-enhancements. First, by Lemma 3.1.11, we have that pretr($\mathbf{A}$) $\subseteq$ h-proj($\mathbf{A}$). Next, consider the functor

$$H^0(\mathrm{pretr}(\mathbf{A})) \subseteq H^0(\text{h-proj}(\mathbf{A})) \xrightarrow{\sim} \mathsf{D}(\mathbf{A}).$$

Since pretr($\mathbf{A}$) is the closure of $\mathbf{A}$ under taking shifts and cones (up to dg-isomorphism), then the above functor restricts to an equivalence

$$H^0(\mathrm{pretr}(\mathbf{A})) \xrightarrow{\sim} \mathrm{tria}(\mathbf{A}), \tag{3.2.3}$$

so, as expected, pretr($\mathbf{A}$) is an enhancement of tria($\mathbf{A}$). Next, let us consider per($\mathbf{A}$). Recall that an object C in a triangulated category $\mathbf{T}$ is *compact* if for any set of objects $T_i \in \mathbf{T}$, the natural map $\bigoplus_i \mathbf{T}(C, T_i) \to \mathbf{T}(C, \bigoplus_i T_i)$ is an isomorphism. We have the following:

[2] A subcategory is called *strictly full* if it is full and closed under isomorphisms.

[3] A triangulated subcategory is called *thick* if it is closed under direct summands.

Lemma 3.2.3 ([Kel06, Corollary 3.7])**.** $\operatorname{per}(\mathbf{A})$ *coincides with the subcategory of compact objects of* $\mathsf{D}(\mathbf{A})$*.*

So, perfect **A**-modules and compact **A**-modules coincide. Now, define $\operatorname{per}_{\mathrm{dg}}(\mathbf{A})$ as the full dg-subcategory of h-proj$(\mathbf{A})$ of the objects which are compact in the category $H^0(\text{h-proj}(\mathbf{A}))$. Then, the fully faithful functor

$$H^0(\operatorname{per}_{\mathrm{dg}}(\mathbf{A})) \hookrightarrow H^0(\text{h-proj}(\mathbf{A})) \xrightarrow{\sim} \mathsf{D}(\mathbf{A})$$

has the compact objects of $\mathsf{D}(\mathbf{A})$ as essential image. So, it restricts to an equivalence

$$H^0(\operatorname{per}_{\mathrm{dg}}(\mathbf{A})) \xrightarrow{\sim} \operatorname{per}(\mathbf{A}), \tag{3.2.4}$$

which proves that $\operatorname{per}_{\mathrm{dg}}(\mathbf{A})$ is an enhancement of $\operatorname{per}(\mathbf{A})$. Since $\operatorname{pretr}(\mathbf{A}) \subseteq \operatorname{per}_{\mathrm{dg}}(\mathbf{A})$, the Yoneda embedding factors through $\operatorname{per}_{\mathrm{dg}}(\mathbf{A})$, yielding a dg-functor

$$\mathbf{A} \hookrightarrow \operatorname{per}_{\mathrm{dg}}(\mathbf{A}). \tag{3.2.5}$$

Recall that a dg-category **A** is pretriangulated if the restricted Yoneda embedding $\mathbf{A} \hookrightarrow \operatorname{pretr}(\mathbf{A})$ is a quasi-equivalence. There is a stronger notion, which is actually more useful:

Definition 3.2.4. Let **A** be a dg-category. We say that **A** is *triangulated*, following Toën, see for instance [Toë11]) if the restricted Yoneda embedding

$$\mathbf{A} \hookrightarrow \operatorname{per}_{\mathrm{dg}}(\mathbf{A})$$

is a quasi-equivalence.

Remark 3.2.5. We see immediately that **A** is pretriangulated if and only if the restricted derived Yoneda embedding

$$H^0(\mathbf{A}) \hookrightarrow \operatorname{tria}(\mathbf{A})$$

is an equivalence; analogously, **A** is triangulated if and only if the restricted derived Yoneda embedding

$$H^0(\mathbf{A}) \hookrightarrow \operatorname{per}(\mathbf{A})$$

is an equivalence. So, **A** is pretriangulated if and only if $H^0(\mathbf{A})$ is a triangulated subcategory of $\mathsf{D}(\mathbf{A})$, and it is triangulated if and only if it is pretriangulated and it is closed under direct summands in $\mathsf{D}(\mathbf{A})$. It is worth mentioning that, equivalently, **A** is triangulated if and only if $H^0(\mathbf{A})$ is an idempotent complete triangulated subcategory of $\mathsf{D}(\mathbf{A})$ (this is because $\operatorname{per}(\mathbf{A})$ is the idempotent completion of $\operatorname{tria}(\mathbf{A})$). So, if an idempotent complete triangulated category has an enhancement, this enhancement is automatically triangulated.

The dg-category $\operatorname{per}_{\mathrm{dg}}(\mathbf{A})$ will be sometimes called the *triangulated hull* of **A**. As we will see at the end of Section 3.5, it satisfies a universal property at the homotopy level. It is worth mentioning that $\operatorname{per}_{\mathrm{dg}}(\mathbf{A})$, as $\operatorname{pretr}(\mathbf{A})$, is invariant under quasi-equivalences. More precisely:

Proposition 3.2.6 ([BLL04, Lemma 4.14])**.** *Let $F\colon \mathbf{A} \to \mathbf{B}$ be a dg-functor. Then, the induction dg-functor Ind_F preserves triangulated hulls, yielding:*

$$\operatorname{per}_{\mathrm{dg}}(F)\colon \operatorname{per}_{\mathrm{dg}}(\mathbf{A}) \to \operatorname{per}_{\mathrm{dg}}(\mathbf{B}).$$

Moreover, if F is a quasi-equivalence, then also $\operatorname{per}_{\mathrm{dg}}(F)$ is such.

In many situations of interest, dg-categories are obtained as triangulated hulls of simpler dg-categories. We are going to make this precise, with the notion of *generators*. Recall that a triangulated category $\mathbf{T}$ is *generated*[4] by a full subcategory $\mathbb{E}$ if $\mathbf{T}$ is the smallest strictly full subcategory of itself which contains $\mathbb{E}$; moreover, $\mathbf{T}$ is *classically generated* by $\mathbb{E}$ if $\mathbf{T}$ is the smallest thick triangulated subcategory of itself which contains $\mathbb{E}$. Clearly, if $\mathbb{E}$ generates $\mathbf{T}$, then it classically generates $\mathbf{T}$.

Definition 3.2.7. Let $\mathbf{A}$ be a triangulated dg-category, and let $\mathbb{E} \subset \mathbf{A}$ be a full dg-subcategory. We say that $\mathbf{A}$ is *generated* by $\mathbb{E}$ if $H^0(\mathbf{A})$ is classically generated by $H^0(\mathbb{E})$.

Remark 3.2.8. The definition of generators of a triangulated dg-category is well behaved. In fact, by [LO10, Proposition 1.16] (keep Corollary 2.3.17 in mind when applying it), we see that if $\mathbb{E}$ generates $\mathbf{A}$, then $\mathbf{A}$ is quasi-equivalent to $\operatorname{per}_{\mathrm{dg}}(\mathbb{E})$. Conversely, if $\mathbf{A} \overset{\mathrm{qe}}{\approx} \operatorname{per}_{\mathrm{dg}}(\mathbb{E})$, then clearly $\mathbf{A}$ is generated by a full dg-subcategory quasi-equivalent to $\mathbb{E}$.

3.3 Derived Isbell duality

In this section, we study a duality result between dg-modules (and also bimodules) which is a vast generalisation of the duality of vector spaces over a field. It is called *Isbell duality*, after John Isbell (see [Woo82] for a reference). Our notation here follows the one found on the nLab[5].

Proposition 3.3.1 (Isbell duality)**.** *Let $\mathbf{A}$ be a dg-category. There is a dg-adjunction*

$$\mathcal{O} \dashv \operatorname{Spec}\colon \mathsf{C}_{\mathrm{dg}}(\mathbf{A}) \leftrightarrows \mathsf{C}_{\mathrm{dg}}(\mathbf{A}^{\mathrm{op}})^{\mathrm{op}}, \tag{3.3.1}$$

where $\mathcal{O}$ and Spec *are defined as follows:*

$$\begin{aligned} \mathcal{O}(X)_A &= \mathsf{C}_{\mathrm{dg}}(\mathbf{A})(X, h_A), \\ \operatorname{Spec}(M)^A &= \mathsf{C}_{\mathrm{dg}}(\mathbf{A}^{\mathrm{op}})(M, h^A). \end{aligned}$$

Proof. We have to prove that there is a natural isomorphism of complexes:

$$\mathsf{C}_{\mathrm{dg}}(\mathbf{A}^{\mathrm{op}})(M, \mathcal{O}(X)) \cong \mathsf{C}_{\mathrm{dg}}(\mathbf{A})(X, \operatorname{Spec}(M)). \tag{3.3.2}$$

[4] We warn the reader for potential confusion of terminology in literature. Some authors (for example, see [BLL04]) give a different meaning to the word "generated".

[5] `ncatlab.org/nlab/show/Isbell+duality`

We compute:

$$\begin{aligned}\mathsf{C}_{\mathrm{dg}}(\mathbf{A}^{\mathrm{op}})(M, \mathcal{O}(X)) &\cong \int_A \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(M_A, \mathcal{O}(X)_A) \\ &= \int_A \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(M_A, \mathsf{C}_{\mathrm{dg}}(\mathbf{A})(X, h_A)) \\ &= \int_A \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(M_A, \int_{A'} \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(X^{A'}, h_A^{A'}))\end{aligned}$$

$$\begin{aligned}&\cong \int_A \int_{A'} \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(M_A \otimes X^{A'}, h_A^{A'}) \\ &\cong \int_{A'} \int_A \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(X^{A'}, \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(M_A, h_A^{A'})) \\ &\cong \int_{A'} \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(X^{A'}, \mathsf{C}_{\mathrm{dg}}(\mathbf{A}^{\mathrm{op}})(M, h^{A'})) \\ &\cong \mathsf{C}_{\mathrm{dg}}(\mathbf{A})(X, \operatorname{Spec}(M)).\end{aligned}$$ □

$\mathcal{O}$ and Spec admit derived functors, by Proposition 3.1.18. So, we obtain the left derived functor

$$\begin{aligned}&\mathbb{L}\mathcal{O} \colon \mathsf{D}(\mathbf{A}) \to \mathsf{D}(\mathbf{A}^{\mathrm{op}})^{\mathrm{op}}, \\ &\mathbb{L}\mathcal{O}(X) = \mathcal{O}(Q(X)).\end{aligned} \tag{3.3.3}$$

Analogously, Spec induces the right derived functor

$$\begin{aligned}&\mathbb{R}\operatorname{Spec} \colon \mathsf{D}(\mathbf{A}^{\mathrm{op}})^{\mathrm{op}} \to \mathsf{D}(\mathbf{A}), \\ &\mathbb{R}\operatorname{Spec}(M) = \operatorname{Spec}(Q(M)).\end{aligned} \tag{3.3.4}$$

Notice that we employed the h-projective resolution even for $\mathbb{R}\operatorname{Spec}$, because of contravariance. By Proposition 3.1.20, we get the derived adjunction

$$\mathbb{L}\mathcal{O} \dashv \mathbb{R}\operatorname{Spec} \colon \mathsf{D}(\mathbf{A}) \to \mathsf{D}(\mathbf{A}^{\mathrm{op}})^{\mathrm{op}}, \tag{3.3.5}$$

which we call *derived Isbell duality.*

An object $X \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A})$ is called *Isbell autodual* if the unit $X \to \operatorname{Spec}(\mathcal{O}(X))$ is an isomorphism. If $X = h_A$ is represented by $A \in \mathbf{A}$, then

$$\begin{aligned}\mathcal{O}(X) &= \mathcal{O}(h_A) \\ &= \mathsf{C}_{\mathrm{dg}}(\mathbf{A})(h_A, h_-) \\ &\cong \mathbf{A}(A, -) = h^A,\end{aligned}$$

and analogously

$$\begin{aligned}\operatorname{Spec}(h^A) &= \mathsf{C}_{\mathrm{dg}}(\mathbf{A}^{\mathrm{op}})(h^A, h^-) \\ &\cong \mathbf{A}(-, A) = h_A.\end{aligned}$$

In the end, we have isomorphisms

$$\begin{aligned} h_A &\cong \operatorname{Spec} \mathcal{O}(h_A), \\ h^A &\cong \mathcal{O}(\operatorname{Spec}(h^A)), \end{aligned}$$

natural in $A \in \mathbf{A}$. By Proposition 1.2.14, we deduce that $\mathbf{A}$-dg-modules of the form h_A are Isbell autodual, and also, more precisely:

Lemma 3.3.2. *The dg-adjunction $\mathcal{O} \dashv \operatorname{Spec}$ restricts to an adjoint dg-equivalence*

$$\operatorname{rep}(\mathbf{A}) \leftrightarrows \operatorname{rep}(\mathbf{A}^{\mathrm{op}})^{\mathrm{op}},$$

where $\operatorname{rep}(\mathbf{A})$ denotes the dg-category of representable right $\mathbf{A}$-modules.

Analogously, the induced adjunction $H^0(\mathcal{O}) \dashv H^0(\operatorname{Spec})$ restricts to an adjoint equivalence

$$\operatorname{hrep}(\mathbf{A}) \leftrightarrows \operatorname{hrep}(\mathbf{A}^{\mathrm{op}})^{\mathrm{op}},$$

where $\operatorname{hrep}(\mathbf{A})$ denotes the full subcategory of $\mathsf{K}(\mathbf{A})$ of $\mathbf{A}$-modules X such that $X \approx h_A$ for some $A \in \mathbf{A}$.

With a little more work, we are able to establish a similar result for the derived adjunction $\mathbb{L}\mathcal{O} \dashv \mathbb{R}\operatorname{Spec}$:

Proposition 3.3.3. *The adjunction $\mathbb{L}\mathcal{O} \dashv \mathbb{R}\operatorname{Spec}$ restricts to an adjoint equivalence*

$$\operatorname{qrep}(\mathbf{A}) \leftrightarrows \operatorname{qrep}(\mathbf{A}^{\mathrm{op}})^{\mathrm{op}},$$

where $\operatorname{qrep}(\mathbf{A})$ is the full subcategory of $\mathsf{D}(\mathbf{A})$ of quasi-representable $\mathbf{A}$-modules*: $X \in \operatorname{qrep}(\mathbf{A})$ if and only if X is quasi-isomorphic to h_A for some $A \in \mathbf{A}$.*

Proof. Let $A \in \mathbf{A}$. Then:

$$\begin{aligned} \mathbb{L}\mathcal{O}(h_A) &= \mathcal{O}(Q(h_A)) \\ &= \mathsf{C}_{\mathrm{dg}}(\mathbf{A})(Q(h_A), h_-) \\ &\overset{\mathrm{qis}}{\approx} \mathsf{C}_{\mathrm{dg}}(\mathbf{A})(h_A, h_-) \\ &\cong h^A, \end{aligned}$$

and analogously $\mathbb{R}\operatorname{Spec}(h^A) \overset{\mathrm{qis}}{\approx} h_A$. The quasi-isomorphism $\mathsf{C}_{\mathrm{dg}}(\mathbf{A})(Q(h_A), h_-) \overset{\mathrm{qis}}{\approx} \mathsf{C}_{\mathrm{dg}}(\mathbf{A})(h_A, h_-)$ induced by $q \colon Q(h_A) \to h_A$ comes from the fact that both $Q(h_A)$ and h_A are h-projective; it is actually a homotopy equivalence. Since q is natural in $\mathsf{K}(\mathbf{A})$, we deduce that $\mathbb{L}\mathcal{O}(h_A) \overset{\mathrm{qis}}{\approx} h^A$ and $\mathbb{R}\operatorname{Spec}(h^A) \overset{\mathrm{qis}}{\approx} h_A$ are natural in $A \in H^0(\mathbf{A}) \hookrightarrow \mathsf{D}(\mathbf{A})$. So, we have natural isomorphisms

$$\begin{aligned} h_A &\overset{\mathrm{qis}}{\approx} \mathbb{R}\operatorname{Spec}(\mathbb{L}\mathcal{O}(h_A)), \\ h^A &\overset{\mathrm{qis}}{\approx} \mathbb{L}\mathcal{O}(\mathbb{R}\operatorname{Spec}(h^A)), \end{aligned}$$

and since $\operatorname{qrep}(\mathbf{A})$ is the isomorphism closure of the image of $H^0(\mathbf{A})$ in $\mathsf{D}(\mathbf{A})$, we conclude with the desired claim. $\square$

Duality for bimodules

(Derived) Isbell duality extends to bimodules. First, let us introduce some notation:

$$\begin{aligned}\mathsf{C}_{\mathrm{dg}}(\mathbf{A},\mathbf{B}) &= \mathsf{C}_{\mathrm{dg}}(\mathbf{B}\otimes\mathbf{A}^{\mathrm{op}}),\\ \mathsf{K}(\mathbf{A},\mathbf{B}) &= \mathsf{K}(\mathbf{B}\otimes\mathbf{A}^{\mathrm{op}}),\\ \mathsf{D}(\mathbf{A},\mathbf{B}) &= \mathsf{D}(\mathbf{B}\otimes^{\mathbb{L}}\mathbf{A}^{\mathrm{op}}),\\ \text{h-proj}(\mathbf{A},\mathbf{B}) &= \text{h-proj}(\mathbf{B}\otimes^{\mathbb{L}}\mathbf{A}^{\mathrm{op}}).\end{aligned}$$

These definitions are justified by the observation that a dg-bimodule $F \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A},\mathbf{B})$ (covariant in $\mathbf{A}$, contravariant in $\mathbf{B}$) can be seen as a dg-functor $F\colon \mathbf{A}\to\mathsf{C}_{\mathrm{dg}}(\mathbf{B})$. Also, remember that by Corollary 3.1.12 the dg-category h-proj$(\mathbf{A},\mathbf{B})$ is an enhancement of $\mathsf{D}(\mathbf{A},\mathbf{B})$.

Isbell duality generalises quite directly to the following:

Proposition 3.3.4. *Let $\mathbf{A},\mathbf{B}$ be dg-categories. There is a dg-adjunction*

$$L \dashv R\colon \mathsf{C}_{\mathrm{dg}}(\mathbf{A},\mathbf{B}) \leftrightarrows \mathsf{C}_{\mathrm{dg}}(\mathbf{B},\mathbf{A})^{\mathrm{op}}, \tag{3.3.6}$$

where L and R are defined by

$$\begin{aligned}L(T)^A_B &= \mathcal{O}(T_A)_B = \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_A,h_B),\\ R(S)^B_A &= \mathrm{Spec}(S^A)^B = \mathsf{C}_{\mathrm{dg}}(\mathbf{B}^{\mathrm{op}})(S^A,h^B).\end{aligned} \tag{3.3.7}$$

Proof. We have to prove that there is a natural isomorphism of complexes:

$$\mathsf{C}_{\mathrm{dg}}(\mathbf{B},\mathbf{A})(S,L(T)) \cong \mathsf{C}_{\mathrm{dg}}(\mathbf{A},\mathbf{B})(T,R(S)). \tag{3.3.8}$$

We compute:

$$\begin{aligned}\mathsf{C}_{\mathrm{dg}}(\mathbf{B},\mathbf{A})(S,L(T)) &\cong \int_A \mathsf{C}_{\mathrm{dg}}(\mathbf{B}^{\mathrm{op}})(S^A,\mathcal{O}(T_A))\\ &\cong \int_A \mathsf{C}_{\mathrm{dg}}(\mathbf{B}^{\mathrm{op}})(T_A,\mathrm{Spec}(S^A))\\ &\cong \mathsf{C}_{\mathrm{dg}}(\mathbf{A},\mathbf{B})(T,R(S)),\end{aligned}$$

where the second isomorphism of the chain follows from the Isbell duality isomorphism (3.3.2) of $\mathbf{B}$. □

By Proposition 3.1.18, L and R can be derived, and in the end we obtain the derived adjunction:

$$\begin{aligned}&\mathbb{L}L \dashv \mathbb{R}R\colon \mathsf{D}(\mathbf{A},\mathbf{B}) \leftrightarrows \mathsf{D}(\mathbf{B},\mathbf{A})^{\mathrm{op}},\\ &\quad\mathbb{L}L(T) = L(Q(T)),\\ &\quad\mathbb{R}R(S) = R(Q(S)).\end{aligned} \tag{3.3.9}$$

The above definitions employ h-projective resolutions of bimodules. A bimodule $T \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})$ induces right $\mathbf{B}$-modules T_A and left $\mathbf{A}$-modules T^B, for all $A \in \mathbf{A}$ and $B \in \mathbf{B}$. A very useful result is that an h-projective resolution of T induces componentwise h-projective resolutions of T_A and T^B (for all A and all B), as explained in the following lemma:

Lemma 3.3.5 (see [CS15, Lemma 3.4])**.** *Let $\mathbf{A}, \mathbf{B}$ be h-projective dg-categories. Let $T \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})$ be an h-projective bimodule. Then, for all $A \in \mathbf{A}$, $T_A \in \mathsf{C}_{\mathrm{dg}}(\mathbf{B})$ is h-projective. Analogously, for all $B \in \mathbf{B}$, $T^B \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A}^{\mathrm{op}})$ is h-projective. In particular, if $q \colon Q(T) \to T$ is an h-projective resolution of T, then $q_A \colon Q(T)_A \to T_A$ and $q^B \colon Q(T)^B \to T^B$ are h-projective resolutions respectively of T_A and T^B, for all $A \in \mathbf{A}$ and $B \in \mathbf{B}$. Without loss of generality, we may set $Q(T)_A = Q(T_A)$ and $Q(T)^B = Q(T^B)$.*

The adjunction $L \dashv R$ and its derived version $\mathbb{L}L \dashv \mathbb{R}R$ are strictly related to the (derived) Isbell duality adjunction. indeed, we have the following:

Lemma 3.3.6. *Let $\mathbf{A}, \mathbf{B}$ be dg-categories. Let $T \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})$ and $S \in \mathsf{C}_{\mathrm{dg}}(\mathbf{B}, \mathbf{A})$. Let $\eta \colon T \to RL(T)$ and $\varepsilon \colon S \to LR(S)$[6] be the unit and counit morphisms of the adjunction $L \vdash R$, calculated in T and S. Then, for all $A \in \mathbf{A}$, the morphisms $\eta_A \colon T_A \to RL(T)_A$ and $\varepsilon_A \colon S^A \to LR(S)^A$ are the unit and counit maps of the Isbell duality of $\mathbf{B}$, calculated in T_A and S^A.*

Proof. We rewrite the adjunction $L \dashv R$ as follows:

$$\int_A \mathsf{C}_{\mathrm{dg}}(\mathbf{B}^{\mathrm{op}})(S^A, L(T)^A) \xrightarrow{\sim} \int_A \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_A, R(S)_A).$$

By definition, $L(T)^A = \mathcal{O}(T_A)$, $R(S)_A = \mathrm{Spec}(S^A)$, and there is a commutative diagram for all $A \in \mathbf{A}$:

$$\begin{array}{ccc} \int_A \mathsf{C}_{\mathrm{dg}}(\mathbf{B}^{\mathrm{op}})(S^A, L(T)^A) & \xrightarrow{\sim} & \int_A \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_A, R(S)_A) \\ \downarrow & & \downarrow \\ \mathsf{C}_{\mathrm{dg}}(\mathbf{B}^{\mathrm{op}})(S^A, L(T)^A) & \xrightarrow{\sim} & \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_A, R(S)_A). \end{array}$$

The vertical arrows are the natural maps associated to the written ends; the "downstairs" isomorphism is precisely the Isbell duality adjunction of $\mathbf{B}$, and our claim immediately follows. □

The above result immediately extends to the homotopy level adjunction $H^0(L) \dashv H^0(R)$, and also to the derived adjunction $\mathbb{L}L \dashv \mathbb{R}R$:

[6]We view the counit as a map in $\mathsf{C}_{\mathrm{dg}}(\mathbf{B}, \mathbf{A})$: this explains the seemingly "wrong direction" of the arrow.

Corollary 3.3.7. *Let* $\mathbf{A}, \mathbf{B}$ *be dg-categories, and let* $T \in \mathsf{D}(\mathbf{A}, \mathbf{B}), S \in \mathsf{D}(\mathbf{B}, \mathbf{A})$. *Let* $\widetilde{\eta}\colon T \to \mathbb{R}R(\mathbb{L}L(T))$ *and* $\widetilde{\varepsilon}\colon S \to \mathbb{L}L(\mathbb{R}R(S))$ *be the unit and counit morphisms of the derived adjunction* $\mathbb{L}L \dashv \mathbb{R}R$*, calculated in* T *and* S*. Then, for all* $A \in \mathbf{A}$*, the morphisms* $\widetilde{\eta}_A\colon T_A \to \mathbb{R}R(\mathbb{L}L(T))_A$ *and* $\widetilde{\varepsilon}_A\colon S^A \to \mathbb{L}L(\mathbb{R}R(S))^A$ *are the unit and counit morphisms of the derived Isbell duality of* $\mathbf{B}$*, calculated in* T_A *and* S^A*.*

Proof. For simplicity, assume that $\mathbf{A}$ and $\mathbf{B}$ are h-projective, identifying them with their h-projective resolutions. Let $A \in \mathbf{A}$. There is an obvious dg-functor

$$\begin{aligned}(-)_A\colon \mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B}) &\to \mathsf{C}_{\mathrm{dg}}(\mathbf{B}),\\ T &\mapsto T_A.\end{aligned}$$

This dg-functor clearly preserves acyclic modules, hence it induces a functor

$$(-)_A\colon \mathsf{D}(\mathbf{A}, \mathbf{B}) \to \mathsf{D}(\mathbf{B}).$$

Recall that, by Lemma 3.3.5, if $q\colon Q(T) \to T$ is an h-projective resolution of T, then $q_A\colon Q(T)_A = Q(T_A) \to T_A$ is an h-projective resolution of T_A. The functor Q is left adjoint to the localisation functor; recalling how this adjunction is obtained (formula (3.1.6)), we see that the diagram

$$\begin{array}{ccc}\mathsf{K}(\mathbf{A}, \mathbf{B})(Q(T), T') & \xrightarrow{\sim} & \mathsf{D}(\mathbf{A}, \mathbf{B})(T, T') \\ \big\downarrow{\scriptstyle (-)_A} & & \big\downarrow{\scriptstyle (-)_A} \\ \mathsf{K}(\mathbf{B})(Q(T_A), T'_A) & \xrightarrow{\sim} & \mathsf{D}(\mathbf{B})(T_A, T'_A).\end{array}$$

is commutative. This, combined with the above lemma and with the definition of the adjunction $\mathbb{L}L \dashv \mathbb{R}R$ as a composition of adjunctions (Proposition 3.1.20), gives us the claim regarding the unit $\widetilde{\eta}$. A similar argument gives the other part of the statement. □

Now, let $T \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})$ be a *right representable* bimodule, that is, for all $A \in \mathbf{A}$, $T_A \cong h_{F(A)}$ for some $F(A) \in \mathbf{B}$. Then, we have that

$$\begin{aligned}L(T)^A &= \mathcal{O}(T_A)\\ &\cong \mathcal{O}(h_{F(A)})\\ &\cong h^{F(A)}.\end{aligned}$$

So, $L(T)$ is *left representable.* Analogously, if $S \in \mathsf{C}_{\mathrm{dg}}(\mathbf{B}, \mathbf{A})$ is left representable, that is, $S^A \cong h^{G(A)}$ for all $A \in \mathbf{A}$, then $R(S)$ is right representable, and in particular $R(S)_A \cong h_{G(A)}$ for all A. So, the duality $L \dashv R$ sends right representables to left representables, and vice-versa. A similar observation can be done at the homotopy level: call a bimodule $T \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})$ *right homotopy representable* if $T_A \approx h_{F(A)}$ for some $F(A) \in \mathbf{B}$, for alla $A \in \mathbf{A}$. Then, a similar computation as above shows that $L(T)^A \approx h^{F(A)}$, so that $L(T)$ is *left homotopy representable.* Vice-versa, if $S \in \mathsf{C}_{\mathrm{dg}}(\mathbf{B}, \mathbf{A})$ is left homotopy representable, then $R(S)$ is right homotopy representable. More precisely, we have the following:

Lemma 3.3.8. *Let* $\mathbf{A}, \mathbf{B}$ *be dg-categories. The dg-adjunction* $L \dashv R$ *restricts to an adjoint dg-equivalence*

$$\operatorname{rep}^r(\mathbf{A}, \mathbf{B}) \leftrightarrows \operatorname{rep}^l(\mathbf{B}, \mathbf{A})^{\mathrm{op}},$$

where $\operatorname{rep}^r(\mathbf{A}, \mathbf{B})$ *is the full dg-subcategory of right representable bimodules in* $\mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})$*, and* $\operatorname{rep}^l(\mathbf{B}, \mathbf{A})$ *is the full dg-subcategory of left representable bimodules in* $\mathsf{C}_{\mathrm{dg}}(\mathbf{B}, \mathbf{A})$*.*

Analogously, the homotopy adjunction $H^0(L) \dashv H^0(R)$ *restricts to an adjoint equivalence*

$$\operatorname{hrep}^r(\mathbf{A}, \mathbf{B}) \leftrightarrows \operatorname{hrep}^l(\mathbf{B}, \mathbf{A})^{\mathrm{op}},$$

where $\operatorname{hrep}^r(\mathbf{A}, \mathbf{B})$ *and* $\operatorname{hrep}^l(\mathbf{B}, \mathbf{A})$ *denote respectively the full subcategories of* $\mathsf{K}(\mathbf{A}, \mathbf{B})$ *and* $\mathsf{K}(\mathbf{B}, \mathbf{A})$ *of* right (or left) homotopy representable bimodules.

Proof. This is a direct application of Lemma 3.3.6. For instance, to show that the unit $\eta \colon T \to RL(T)$ is an isomorphism when $T \in \operatorname{rep}^r(\mathbf{A}, \mathbf{B})$, or a homotopy equivalence when $T \in \operatorname{hrep}^r(\mathbf{A}, \mathbf{B})$, it is sufficient to show that the components $\eta_A \colon T_A \to RL(T)_A$ are such for all $A \in \mathbf{A}$. But by hypothesis $T_A \in \operatorname{rep}(\mathbf{B})$ (or $\operatorname{hrep}(\mathbf{B})$ in the case of homotopy right representability), so by Lemma 3.3.2 we are done. □

A similar result as above holds for the derived duality $\mathbb{L}L \dashv \mathbb{R}R$. Call a bimodule $T \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})$ *right quasi-representable* if $T_A \overset{\mathrm{qis}}{\approx} h_{F(A)}$ for some $F(A) \in \mathbf{B}$, for all $A \in \mathbf{A}$; analogously, a bimodule $S \in \mathsf{C}_{\mathrm{dg}}(\mathbf{B}, \mathbf{A})$ is called *left quasi-representable* if $S^A \overset{\mathrm{qis}}{\approx} h^{G(A)}$ for some $G(A) \in \mathbf{B}$, for all $A \in \mathbf{A}$. We have the following:

Proposition 3.3.9. *Let* $\mathbf{A}, \mathbf{B}$ *be dg-categories. The derived adjunction* $\mathbb{L}L \dashv \mathbb{R}R$ *restricts to an adjoint equivalence*

$$\operatorname{qrep}^r(\mathbf{A}, \mathbf{B}) \leftrightarrows \operatorname{qrep}^l(\mathbf{B}, \mathbf{A})^{\mathrm{op}},$$

where $\operatorname{qrep}^r(\mathbf{A}, \mathbf{B})$ *is the full subcategory of* $\mathsf{D}(\mathbf{A}, \mathbf{B})$ *of right quasi-representable bimodules, and* $\operatorname{qrep}^l(\mathbf{B}, \mathbf{A})$ *is the full subcategory of* $\mathsf{D}(\mathbf{B}, \mathbf{A})$ *of left quasi-representable bimodules.*

Proof. This is an application of Corollary 3.3.7. For instance, to show that the unit $\widetilde{\eta} \colon T \to \mathbb{R}R(\mathbb{L}L(T))$ is an isomorphism in $\mathsf{D}(\mathbf{A}, \mathbf{B})$, it is sufficient to show that $\widetilde{\eta}_A$ is an isomorphism in $\mathsf{D}(\mathbf{B})$ for all A. This follows directly by Proposition 3.3.3, since by hypothesis $T_A \in \operatorname{qrep}(\mathbf{B})$ for all $A \in \mathbf{A}$. □

3.4 The bicategory of bimodules; adjoints

An interesting feature of bimodules is that they can be viewed as "generalised functors". We will sometimes write $F\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ meaning $F \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})$. Given bimodules $F\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ and $G\colon \mathbf{B} \rightsquigarrow \mathbf{C}$, we can define their composition $G \diamond F\colon \mathbf{A} \rightsquigarrow \mathbf{C}$, as follows:

$$(G \diamond F)_A^C = \int^B F_A^B \otimes G_B^C. \tag{3.4.1}$$

Applying the dg-functoriality of coends, we find out that $\diamond$ is dg-functorial in both variables, hence giving rise to dg-bifunctors

$$- \diamond -\colon \mathsf{C}_{\mathrm{dg}}(\mathbf{B}, \mathbf{C}) \otimes \mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B}) \to \mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{C}). \tag{3.4.2}$$

In particular, if $\varphi\colon F \to F'$ and $\psi\colon G \to G'$ are dg-natural transformations, we have dg-natural transformations $\psi \diamond F\colon G \diamond F \to G' \diamond F$ and $G \diamond \varphi\colon G \diamond F \to G \diamond F'$.

By co-Yoneda lemma, the diagonal bimodules act as (weak) units for this composition:

$$\begin{aligned} F \diamond h_{\mathbf{A}} &= \int^A h^A \otimes F_A \cong F, \\ h_{\mathbf{B}} \diamond F &= \int^B F^B \otimes h_B \cong F, \end{aligned} \tag{3.4.3}$$

given $F\colon \mathbf{A} \rightsquigarrow \mathbf{B}$. Moreover, the composition is weakly associative. indeed, given $F\colon \mathbf{A} \to \mathbf{B}, G\colon \mathbf{B} \to \mathbf{C}, H\colon \mathbf{C} \to \mathbf{D}$, we have:

$$\begin{aligned} H \diamond (G \diamond F) &= \int^C (G \diamond F)^C \otimes H_C \\ &= \int^C \left(\int^B F^B \otimes G_B^C \right) \otimes H_C \\ &\cong \int^B \int^C F^B \otimes (G_B^C \otimes H_C) \\ &\cong \int^B F^B \otimes \left(\int^C G_B^C \otimes H_C \right) \\ &= \int^B F^B \otimes (H \diamond G)_B \\ &= (H \diamond G) \diamond F, \end{aligned}$$

where we used Fubini's theorem and the cocontinuity of the tensor product.

Another interesting property of the composition $\diamond$ is that it preserves h-projective bimodules:

Lemma 3.4.1. *Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be h-projective dg-categories. Let $F\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ and $G\colon \mathbf{B} \rightsquigarrow \mathbf{C}$ be h-projective bimodules. Then, $G \diamond F$ is h-projective.*

Proof. Let $N \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{C})$ be acyclic. We compute:

$$\begin{aligned}
\mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{C})(G \diamond F, N) &= \int_{A,C} \mathsf{C}_{\mathrm{dg}}(\mathbf{k})((G \diamond F)^C_A, N^C_A) \\
&= \int_{A,C} \mathsf{C}_{\mathrm{dg}}(\mathbf{k}) \left(\int^B F^B_A \otimes G^C_B, N^C_A \right) \\
&\cong \int_{A,B,C} \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(F^B_A \otimes G^C_B, N^C_A) \\
&\cong \int_{A,B,C} \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(F^B_A, \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(G^C_B, N^C_A)) \\
&\cong \int_{A,B} \mathsf{C}_{\mathrm{dg}}(\mathbf{k})(F^B_A, \mathsf{C}_{\mathrm{dg}}(\mathbf{C})(G_B, N_A)) \\
&\cong \mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})(F, \mathsf{C}_{\mathrm{dg}}(\mathbf{C})(G_-, N_-)).
\end{aligned}$$

Now, $(A, B) \mapsto \mathsf{C}_{\mathrm{dg}}(\mathbf{C})(G_B, N_A)$ is acyclic, indeed:

$$\begin{aligned}
H^i(\mathsf{C}_{\mathrm{dg}}(\mathbf{C})(G_B, N_A)) &= H^0(\mathsf{C}_{\mathrm{dg}}(\mathbf{C})(G_B, N_A[i]) \\
&= \mathsf{K}(\mathbf{C})(G_B, N_A[i]),
\end{aligned}$$

and G_B is h-projective by Lemma 3.3.5, so $\mathsf{K}(\mathbf{C})(G_B, N_A[i]) \cong 0$. Hence, since F is h-projective, $\mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})(F, \mathsf{C}_{\mathrm{dg}}(\mathbf{C})(G_-, N_-))$ is acyclic, and we are done. □

There is a derived version of the composition $\diamond$. Namely, given $F \in \mathsf{D}(\mathbf{A}, \mathbf{B})$ and $G \in \mathsf{D}(\mathbf{B}, \mathbf{C})$, we set

$$G \diamond^{\mathbb{L}} F = Q(G) \diamond Q(F) \overset{\text{qis}}{\approx} G \diamond Q(F) \overset{\text{qis}}{\approx} Q(F) \diamond G \tag{3.4.4}$$

taking h-projective resolutions of either F or G. The composition $\diamond^{\mathbb{L}}$ is defined up to quasi-isomorphism, and it is functorial, as we can expect, in the sense that it gives bifunctors

$$- \diamond^{\mathbb{L}} - \colon \mathsf{D}(\mathbf{B}, \mathbf{C}) \otimes \mathsf{D}(\mathbf{A}, \mathbf{B}) \to \mathsf{D}(\mathbf{A}, \mathbf{C}). \tag{3.4.5}$$

By the above Lemma 3.4.1, $Q(G) \diamond Q(F)$ is always h-projective, so we may prove directly that $\diamond^{\mathbb{L}}$ is weakly associative and unital. indeed:

$$\begin{aligned}
H \diamond^{\mathbb{L}} (G \diamond^{\mathbb{L}} F) &= Q(H) \diamond (G \diamond^{\mathbb{L}} F) \\
&= Q(H) \diamond (Q(G) \diamond Q(F)) \\
&\cong (Q(H) \diamond Q(G)) \diamond Q(F) \\
&= (H \diamond^{\mathbb{L}} G) \diamond Q(F) \\
&= (H \diamond^{\mathbb{L}} G) \diamond^{\mathbb{L}} F,
\end{aligned}$$

and

$$F \diamond^{\mathbb{L}} h_{\mathbf{A}} = Q(F) \diamond h_{\mathbf{A}} \cong Q(F) \overset{\text{qis}}{\approx} F,$$

$$h_{\mathbf{B}} \diamond^{\mathbb{L}} F = h_{\mathbf{B}} \diamond Q(F) \cong Q(F) \overset{\text{qis}}{\approx} F.$$

The composition $\diamond$ and its derived version $\diamond^{\mathbb{L}}$ are indeed part of *bicategorical* structures. Namely, we have the (dg-)bicategory Bimod whose objects are dg-categories, with 1-morphisms and 2-morphisms respectively given by the objects and the morphisms of the dg-categories $\mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})$; in the derived setting, we have the bicategory DBimod whose objects are dg-categories, with 1-morphisms and 2-morphisms given respectively by the objects and the morphisms of the categories $\mathsf{D}(\mathbf{A}, \mathbf{B})$. We won't study these structures in full detail; however, it is interesting to explore the notion of adjointness and its relation to (quasi)-representability.

Definition 3.4.2. Let $F \colon \mathbf{A} \rightsquigarrow \mathbf{B}$ and $G \colon \mathbf{B} \rightsquigarrow \mathbf{A}$ be 1-morphisms in Bimod. We say that $F \dashv G$ (F is left adjoint to G) if there exist (closed, degree 0) 2-morphisms $\eta \colon h_{\mathbf{A}} \to G \diamond F$ and $\varepsilon \colon F \diamond G \to h_{\mathbf{B}}$ such that the following triangular identities are satisfied:

$$(F \cong F \diamond h_{\mathbf{A}} \xrightarrow{F \diamond \eta} F \diamond (G \diamond F) \cong (F \diamond G) \diamond F \xrightarrow{\varepsilon \diamond F} h_{\mathbf{B}} \diamond F \cong F) = 1_F,$$
$$(G \cong h_{\mathbf{A}} \diamond G \xrightarrow{\eta \diamond G} (G \diamond F) \diamond G \cong G \diamond (F \diamond G) \xrightarrow{G \diamond \varepsilon} G \diamond h_{\mathbf{B}} \cong G) = 1_G.$$

The definition of adjoint 1-morphisms in DBimod is analogous (replace $\diamond$ with the derived composition $\diamond^{\mathbb{L}}$).

Given $T \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})$, we could expect that its dual $L(T)$ (or $R(T)$) were adjoint to T. This is not true in general, but it is provable under the right (or the left) representability assumption. First, notice that there is a (closed, degree 0) morphism in $\mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{A})$, dg-natural in T:

$$n \colon L(T) \diamond T \to \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_-, T_-). \tag{3.4.6}$$

Indeed, write:

$$\begin{aligned}
(L(T) \diamond T)^A_{A'} &= \int^B T^B_{A'} \otimes L(T)^A_B \\
&= \int^B T^B_{A'} \otimes \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_A, h_B) \\
&\cong \int^B \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(h_B, T_{A'}) \otimes \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_A, h_B).
\end{aligned}$$

It is directly checked that the diagram

$$\begin{array}{ccc}
\mathsf{C}_{\mathrm{dg}}(\mathbf{B})(h_{B'}, T_{A'}) \otimes \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_A, h_B) & \longrightarrow & \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(h_B, T_{A'}) \otimes \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_A, h_B) \\
\downarrow & & \downarrow \\
\mathsf{C}_{\mathrm{dg}}(\mathbf{B})(h_{B'}, T_{A'}) \otimes \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_A, h_{B'}) & \longrightarrow & \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_A, T_{A'})
\end{array}$$

is commutative for all $B \to B'$ in $\mathbf{B}$, where the arrows arriving in $\mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_A, T_{A'})$ are given by composition, and they are natural in A, A'. Hence, by the universal property of the coend, we find our desired map. There are also (closed, degree 0) maps

$$e \colon T \diamond \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_-, T_-) \to T, \tag{3.4.7}$$
$$e' \colon \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_-, T_-) \diamond L(T) \to L(T). \tag{3.4.8}$$

The morphism (3.4.7) is induced by the maps

$$\begin{aligned} \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_A, T_{A'}) \otimes T_A^B &\to T_{A'}^B, \\ \varphi \otimes x &\mapsto \varphi^B(x), \end{aligned}$$

natural in A' and B. Moreover, (3.4.8) is induced by the composition maps

$$\mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_A, h_B) \otimes \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_{A'}, T_A) \to \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_{A'}, h_B),$$

natural in A' and B. In a similar fashion as for (3.4.7), we get a candidate counit morphism:

$$\varepsilon \colon T \diamond L(T) \to h_{\mathbf{B}}. \tag{3.4.9}$$

indeed, this morphism is induced by the maps:

$$\begin{aligned} \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_A, h_{B'}) \otimes T_A^B &\to h_{B'}^B, \\ \varphi \otimes x &\mapsto \varphi^B(x), \end{aligned}$$

natural in B and B'. Also, we have the morphism

$$t \colon h_{\mathbf{A}} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_-, T_-),$$

induced by the action of T on morphisms of $\mathbf{A}$. The following result tells us that the adjunction $T \dashv L(T)$ is not very far from being obtained.

Lemma 3.4.3. *The diagram*

$$\begin{array}{ccccccc} T & \xrightarrow{\sim} & T \diamond h_{\mathbf{A}} & \xrightarrow{T \diamond t} & T \diamond \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_-, T_-) & \xrightarrow{e} & T \\ & & & \nearrow{\scriptstyle T \diamond n} & & & \uparrow{\scriptstyle \sim} \\ & & T \diamond (L(T) \diamond T) & \xrightarrow{\sim} & (T \diamond L(T)) \diamond T & \xrightarrow{\varepsilon \diamond T} & h_{\mathbf{B}} \diamond T \end{array} \tag{3.4.10}$$

is commutative, and the top row composition is the identity 1_T.

Analogously, the diagram

$$\begin{array}{ccccccc} L(T) & \xrightarrow{\sim} & h_{\mathbf{A}} \diamond L(T) & \xrightarrow{t \diamond L(T)} & \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_-, T_-) \diamond L(T) & \xrightarrow{e'} & L(T) \\ & & & \nearrow{\scriptstyle n \diamond L(T)} & & & \uparrow{\scriptstyle \sim} \\ & & (L(T) \diamond T) \diamond L(T) & \xrightarrow{\sim} & L(T) \diamond (T \diamond L(T)) & \xrightarrow{L(T) \diamond \varepsilon} & L(T) \diamond h_{\mathbf{B}} \end{array} \tag{3.4.11}$$

is commutative, and the top row composition is the identity $1_{L(T)}$.

Proof. They are all direct computations, which we leave to the reader. □

Taking h-projective resolutions of T and of $\mathbb{L}L(T)$, and projecting every morphism in the derived category, we get the derived version of the above lemma:

Lemma 3.4.4. *The diagram*

$$
\begin{tikzcd}
T \arrow[r, "\sim"] & T \diamond^{\mathbb{L}} h_{\mathbf{A}} \arrow[r, "T\diamond^{\mathbb{L}} t"] & T \diamond^{\mathbb{L}} \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(Q(T)_-, Q(T)_-) \arrow[r, "e"] & T \\
T \diamond^{\mathbb{L}} (\mathbb{L}L(T) \diamond^{\mathbb{L}} T) \arrow[ur, "T\diamond^{\mathbb{L}} n"] \arrow[r, "\sim"] & (T \diamond^{\mathbb{L}} \mathbb{L}L(T)) \diamond^{\mathbb{L}} T \arrow[r, "\varepsilon\diamond^{\mathbb{L}} T"] & h_{\mathbf{B}} \diamond^{\mathbb{L}} T \arrow[u, "\sim"'] &
\end{tikzcd}
\tag{3.4.12}
$$

is commutative, and the top row composition is the identity 1_T.

Analogously, the diagram

$$
\begin{tikzcd}
\mathbb{L}L(T) \overset{\mathrm{qis}}{\approx} h_{\mathbf{A}} \diamond^{\mathbb{L}} \mathbb{L}L(T) \arrow[r, "t\diamond^{\mathbb{L}} L(T)"] & \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(Q(T)_-, Q(T)_-) \diamond^{\mathbb{L}} \mathbb{L}L(T) \arrow[r, "e'"] & \mathbb{L}L(T) \\
(\mathbb{L}L(T) \diamond^{\mathbb{L}} T) \diamond^{\mathbb{L}} \mathbb{L}L(T) \arrow[ur, "n\diamond^{\mathbb{L}}\mathbb{L}L(T)"] \arrow[r, "\sim"] & \mathbb{L}L(T) \diamond^{\mathbb{L}} (T \diamond^{\mathbb{L}} \mathbb{L}L(T)) \arrow[r, "\mathbb{L}L(T)\diamond^{\mathbb{L}}\varepsilon"] & \mathbb{L}L(T) \diamond^{\mathbb{L}} h_{\mathbf{B}} \arrow[u, "\sim"']
\end{tikzcd}
\tag{3.4.13}
$$

is commutative, and the top row composition is the identity $1_{\mathbb{L}L(T)}$.

Proof. It follows immediately from Lemma 3.4.3. Remember to compose with the h-projective resolutions $T \overset{\mathrm{qis}}{\approx} Q(T)$ and $\mathbb{L}L(T) \overset{\mathrm{qis}}{\approx} Q(\mathbb{L}L(T))$, at the start and the end of the top rows of both diagrams. □

Now, we see that the obstruction for $L(T)$ to be adjoint to T lies in the morphism (3.4.6) (or its derived version). For instance, if it is a natural isomorphism, then we may define the unit morphism

$$\eta = n^{-1}t \colon h_{\mathbf{A}} \to L(T) \diamond T,$$

and Lemma 3.4.3 tells us immediately that $T \dashv L(T)$. Analogously, if the derived morphism $n \colon \mathbb{L}L(T) \diamond^{\mathbb{L}} T \to \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(Q(T)_-, Q(T)_-)$ is a quasi-isomorphism, then we have a unit morphism η in the derived category, and Lemma 3.4.4 tells us that $T \dashv \mathbb{L}L(T)$. A sufficient condition for n to be (in some sense) invertible is actually the right (quasi-)representability of T:

Proposition 3.4.5. *If* $T \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})$ *is right representable, then* (3.4.6) *is an isomorphism. If it is right homotopy representable, then it is a homotopy equivalence.*

If $T \in \mathsf{D}(\mathbf{A}, \mathbf{B})$ *is right quasi-representable, then* (3.4.6) *induces a quasi-isomorphism (that is, the derived map*

$$n \colon \mathbb{L}L(T) \diamond^{\mathbb{L}} T \to \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(Q(T)_-, Q(T)_-)$$

is a quasi-isomorphism).

Proof. Assume that $T_A \cong h_{F(A)}$ or $T_A \approx h_{F(A)}$ for all $A \in \mathbf{A}$. Then, we have a commutative diagram:

$$\begin{array}{ccc}
\int^B T_{A'}^B \otimes \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_A, h_B) & \xrightarrow{\quad n \quad} & \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(T_A, T_{A'}) \\
\big\downarrow \approx & & \big\downarrow \approx \\
\int^B h_{F(A')}^B \otimes \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(h_{F(A)}, h_B) & & \mathsf{C}_{\mathrm{dg}}(\mathbf{B})(h_{F(A)}, h_{F(A')}) \\
\big\downarrow \sim & & \big\downarrow \sim \\
\int^B h_{F(A')}^B \otimes h_B^{F(A)} & \xrightarrow{\quad \sim \quad} & h_{F(A')}^{F(A)}.
\end{array}$$

The lower vertical arrows, labeled with $\sim$, are given by the Yoneda lemma; the lower horizontal arrow is the co-Yoneda isomorphism. By dg-functoriality, the upper vertical arrows, labeled with $\approx$, are isomorphisms if T is right representable, homotopy equivalences if T is homotopy right representable. So, in the first case, n is an isomorphism, and in the other case n is a homotopy equivalence.

In the derived setting, just replace T with its h-projective resolution $Q(T)$. Then, since $Q(T)_A$ and $h_{F(A)}$ are h-projective for all A, the quasi-isomorphism $Q(T)_A \overset{\mathrm{qis}}{\approx} h_{F(A)}$ is actually given by a homotopy equivalence; we apply the above argument and conclude that n is a quasi-isomorphism, when viewed in the derived category. □

Corollary 3.4.6. *If $T \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})$ is right representable, then there is an adjunction $T \dashv L(T)$ in* Bimod*. If $T \in \mathsf{D}(\mathbf{A}, \mathbf{B})$ is right quasi-representable, then there is an adjunction $T \dashv \mathbb{L}L(T)$ in* DBimod*.*

Moreover, if $S \in \mathsf{C}_{\mathrm{dg}}(\mathbf{B}, \mathbf{A})$ is left representable, then there is an adjunction $R(S) \dashv S$ in Bimod*. If $S \in \mathsf{D}(\mathbf{B}, \mathbf{A})$ is left quasi-representable, then there is an adjunction $\mathbb{R}R(S) \dashv S$ in* DBimod*.*

Proof. The first part of the assertion follows directly from Proposition 3.4.5 and the above discussion. The second part is a consequence of Lemma 3.3.2 and Proposition 3.3.3. Indeed, if S is left representable, then $R(S)$ is right representable, so we have $R(S) \dashv LR(S)$, but $LR(S) \cong S$, and we are done. A similar argument in the derived setting shows that $\mathbb{R}R(S) \dashv S$. □

3.5 Quasi-functors

Let $T \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})$ be a right representable bimodule, and assume $T_A \cong h_{F(A)}$ for all $A \in \mathbf{A}$. Then, there is a (unique) way to define a dg-functor $F \colon \mathbf{A} \to \mathbf{B}$ such that the above isomorphisms are natural in A:

$$\begin{array}{ccc} T_A & \xrightarrow{\sim} & h_{F(A)} \\ \downarrow{\scriptstyle T_f} & & \downarrow{\scriptstyle h_{F(f)}} \\ T_{A'} & \xrightarrow{\sim} & h_{F(A')}. \end{array}$$

However, if $T \in \mathsf{D}(\mathbf{A}, \mathbf{B})$ is right quasi-representable, $T_A \overset{\text{qis}}{\approx} h_{F(A)}$, then the above technique fails. Indeed, for all $f\colon A \to A'$ in $\mathbf{A}$, consider the following diagram:

$$\begin{array}{ccc} T_A & \xrightarrow{\overset{\text{qis}}{\approx}} & h_{F(A)} \\ \downarrow{\scriptstyle T_f} & & \vdots \\ T_{A'} & \xrightarrow{\overset{\text{qis}}{\approx}} & h_{F(A')} \end{array}$$

For simplicity, assume that T_A is h-projective for all A, so that the above horizontal quasi-isomorphisms are actually homotopy equivalences. Then, we are led to define $h_{F(f)}\colon h_{F(A)} \to h_{F(A')}$ by choosing a weak inverse of $T_A \approx h_{F(A)}$ and composing with T_f and a representative of $T_{A'} \approx h_{F(A')}$. This gives us an arrow $F(f)\colon F(A) \to F(A')$, but this arrow is not uniquely determined (it is just "unique up to homotopy"), so we are unable to obtain a dg-functor from this, even if we could in fact show that our attempt to define F gives a "weak (homotopy coherent) dg-functor". In fact, right quasi-representable bimodules are themselves higher categorical gadgets: they are called *quasi-functors*.

The category of quasi-functors from $\mathbf{A}$ to $\mathbf{B}$, which we called $\mathrm{qrep}^r(\mathbf{A}, \mathbf{B})$, is usually denoted by $\mathrm{rep}(\mathbf{A}, \mathbf{B})$ in literature (see, for instance, [Kel06]). In order to avoid confusion, we will stick to our (non standard) notation. Often, we will allow ourselves to write $T\colon \mathbf{A} \to \mathbf{B}$ to mean that T is a quasi-functor from $\mathbf{A}$ to $\mathbf{B}$. The composition $\diamond^{\mathbb{L}}$ descends to quasi-functors, namely, if $T \in \mathrm{qrep}^r(\mathbf{A}, \mathbf{B})$ and $S \in \mathrm{qrep}^r(\mathbf{B}, \mathbf{C})$, then $S \diamond^{\mathbb{L}} T \in \mathrm{qrep}^r(\mathbf{A}, \mathbf{C})$. Indeed, assume that $T_A \overset{\text{qis}}{\approx} h_{F(A)}$ and $S_B \overset{\text{qis}}{\approx} h_{G(B)}$ for all $A \in \mathbf{A}$ and $B \in \mathbf{B}$. Then:

$$(S \diamond^{\mathbb{L}} T)_A = \int^B Q(T)^B_A \otimes Q(S)_B \approx \int^B h^B_{F(A)} \otimes h_{G(B)} \cong h_{G(F(A))}, \tag{3.5.1}$$

where the last isomorphism follows by co-Yoneda lemma. It is also worth remarking that any dg-functor $F\colon \mathbf{A} \to \mathbf{B}$ can be identified with a quasi-functor, namely, the bimodule h_F. Moreover, notice that the composition of F with a quasi-functor (actually, any bimodule) $G\colon \mathbf{B} \to \mathbf{C}$ yields the following:

$$\begin{aligned} (G \diamond^{\mathbb{L}} F)_A &= \int^B h^B_{F(A)} \otimes Q(G)_B \\ &\cong Q(G)_{F(A)} \\ &\overset{\text{qis}}{\approx} G_{F(A)}, \end{aligned} \tag{3.5.2}$$

dg-functorially in A, for F is a dg-functor. Clearly, the above computation gives a similar result also for the (underived) composition $\diamond$. Now, corollary 3.1.12 immediately specialises to the following:

Corollary 3.5.1. *The full dg-subcategory* $\text{h-proj}^{rqr}(\mathbf{A}, \mathbf{B}) \subset \text{h-proj}(\mathbf{A}, \mathbf{B})$ *of h-projective right quasi-representable bimodules is an enhancement of* $\operatorname{qrep}^r(\mathbf{A}, \mathbf{B})$*. More precisely, the equivalence*

$$H^0(\text{h-proj}^{rqr}(\mathbf{A}, \mathbf{B})) \xrightarrow{\sim} \operatorname{qrep}^r(\mathbf{A}, \mathbf{B})$$

is obtained from (3.1.7) *by restriction.*

The dg-category $\text{h-proj}^{rqr}(\mathbf{A}, \mathbf{B})$ can be thought as the *dg-category of quasi-functors* from $\mathbf{A}$ to $\mathbf{B}$. By the above computation (3.5.1) and by Lemma 3.4.1, we see that the composition $\diamond$ preserves h-projective right quasi-representable bimodules, and in fact it gives dg-bifunctors

$$- \diamond - \colon \text{h-proj}^{rqr}(\mathbf{B}, \mathbf{C}) \otimes \text{h-proj}^{rqr}(\mathbf{A}, \mathbf{B}) \to \text{h-proj}^{rqr}(\mathbf{A}, \mathbf{C}). \tag{3.5.3}$$

The relevance of $\text{h-proj}^{rqr}(\mathbf{A}, \mathbf{B})$ lies in the fact that it is an incarnation of the internal hom $\mathbb{R}\underline{\operatorname{Hom}}(-,-)$ of the monoidal category Hqe:

Theorem 3.5.2 ([Toë07], [CS15])**.** *There is a natural bijection:*

$$\mathsf{Hqe}(\mathbf{A} \otimes^{\mathbb{L}} \mathbf{B}, \mathbf{C}) \xrightarrow{\sim} \mathsf{Hqe}(\mathbf{A}, \text{h-proj}^{rqr}(\mathbf{B}, \mathbf{C})),$$

which lifts to a natural quasi-equivalence:

$$\text{h-proj}^{rqr}(\mathbf{A} \otimes^{\mathbb{L}} \mathbf{B}, \mathbf{C}) \xrightarrow{\sim} \text{h-proj}^{rqr}(\mathbf{A}, \text{h-proj}^{rqr}(\mathbf{B}, \mathbf{C})).$$

From now until the end of this chapter, we identify the dg-category $\mathbb{R}\underline{\operatorname{Hom}}(\mathbf{A}, \mathbf{B})$ to $\text{h-proj}^{rqr}(\mathbf{A}, \mathbf{B})$, and moreover we identify $H^0(\mathbb{R}\underline{\operatorname{Hom}}(\mathbf{A}, \mathbf{B}))$ to the category of quasi-functors $\operatorname{qrep}^r(\mathbf{A}, \mathbf{B})$. Nevertheless, we mention that we will sometimes employ the term *quasi-functor* from $\mathbf{A}$ to $\mathbf{B}$ to mean an element of $H^0(\mathbb{R}\underline{\operatorname{Hom}}(\mathbf{A}, \mathbf{B}))$, regardless of the chosen "incarnation" of $\mathbb{R}\underline{\operatorname{Hom}}(\mathbf{A}, \mathbf{B})$; moreover, we will often simplify notation and denote the composition of quasi-functors by $\circ$ or even juxtaposition. We conclude the present discussion mentioning the "homotopy universal properties" of the pretriangulated envelope $\operatorname{pretr}(\mathbf{A})$ and the triangulated hull $\operatorname{per}(\mathbf{A})$.

Proposition 3.5.3 ([Kel06, Paragraph 4.5])**.** *Let* $\mathbf{A}, \mathbf{B}$ *be dg-categories, and assume that* $\mathbf{B}$ *is pretriangulated. Then* $\mathbb{R}\underline{\operatorname{Hom}}(\mathbf{A}, \mathbf{B})$ *is pretriangulated. Moreover, there is a natural quasi-equivalence:*

$$\mathbb{R}\underline{\operatorname{Hom}}(\operatorname{pretr}(\mathbf{A}), \mathbf{B}) \xrightarrow{\sim} \mathbb{R}\underline{\operatorname{Hom}}(\mathbf{A}, \mathbf{B}), \tag{3.5.4}$$

induced by the Yoneda embedding $\mathbf{A} \hookrightarrow \operatorname{pretr}(\mathbf{A})$*.*

Proposition 3.5.4 ([Toë07, Theorem 7.2], [CS15, Corollary 4.2])**.** *Let* $\mathbf{A}, \mathbf{B}$ *be dg-categories, and assume that* $\mathbf{B}$ *is triangulated. Then* $\mathbb{R}\underline{\operatorname{Hom}}(\mathbf{A}, \mathbf{B})$ *is triangulated. Moreover, there is a natural quasi-equivalence:*

$$\mathbb{R}\underline{\operatorname{Hom}}(\operatorname{per}_{\mathrm{dg}}(\mathbf{A}), \mathbf{B}) \xrightarrow{\sim} \mathbb{R}\underline{\operatorname{Hom}}(\mathbf{A}, \mathbf{B}), \tag{3.5.5}$$

induced by the Yoneda embedding $\mathbf{A} \hookrightarrow \operatorname{per}_{\mathrm{dg}}(\mathbf{A})$*.*

Adjoints

The results of Section 3.4 allow us to give a simple working characterisation of adjunctions of quasi-functors. Given quasi-functors $T\colon \mathbf{A} \to \mathbf{B}$ and $S\colon \mathbf{B} \to \mathbf{A}$, we say that T *is left adjoint to* S *(and* S *is right adjoint to* T*)* simply if $T \dashv S$ as bimodules, that is, in the bicategory DBimod. Now, since T is a quasi-functor, then we have the adjunction $T \dashv \mathbb{L}L(T)$, and so $S \overset{\text{qis}}{\approx} \mathbb{L}L(T)$ (adjoints are always unique up to isomorphism). In particular, S is left quasi-representable, and we have the adjunction $\mathbb{R}R(S) \dashv S$, so we also deduce that $T \overset{\text{qis}}{\approx} \mathbb{R}R(S)$. In conclusion, we get the following result:

Proposition 3.5.5. *A quasi-functor* $T\colon \mathbf{A} \to \mathbf{B}$ *has a left adjoint if and only if it is left quasi-representable. Moreover, it has a right adjoint if and only if* $\mathbb{L}L(T)$ *is right quasi-representable.*

There are sufficient hypotheses on the dg-categories that guarantee the existence of adjoints. They are, in some sense, particular finiteness conditions:

Definition 3.5.6. Let $\mathbf{A}$ be a dg-category. We say that $\mathbf{A}$ is *locally perfect* if $\mathbf{A}(A, A')$ is a perfect $\mathbf{k}$-module: $\mathbf{A}(A, A') \in \operatorname{per}(\mathbf{k})$ for all $A, A' \in \mathbf{A}$. We say that $\mathbf{A}$ is *smooth* if the diagonal bimodule is perfect: $h_{\mathbf{A}} \in \operatorname{per}(\mathbf{A} \otimes^{\mathbb{L}} \mathbf{A}^{\text{op}})$.

Remark 3.5.7. It is worth mentioning that a dg-category $\mathbf{A}$ is smooth (resp. locally perfect) if and only $\operatorname{per}_{\text{dg}}(\mathbf{A})$ is smooth (resp. locally perfect): see [TV07, Lemma 2.6].

We need a result adapted from [TV07, Lemma 2.8]:

Lemma 3.5.8. *Let* $\mathbf{A}, \mathbf{B}$ *be dg-categories, and let* $T \in \mathsf{D}(\mathbf{A}, \mathbf{B})$ *be a bimodule. If* $\mathbf{A}$ *is locally perfect and* T *is a perfect bimodule, then* T_A *is a perfect right* $\mathbf{B}$*-module for all* $A \in \mathbf{A}$*. Analogously, if* $\mathbf{B}$ *is locally perfect and* T *is perfect, then* T^B *is a perfect left* $\mathbf{A}$*-module for all* $B \in \mathbf{B}$*.*

Conversely, if $\mathbf{A}$ *is smooth and* T_A *is perfect for all* $A \in \mathbf{A}$*, then* T *is a perfect bimodule. Analogously, if* $\mathbf{B}$ *is smooth and* T^B *is perfect for all* $B \in \mathbf{B}$*, then* T *is perfect.*

Now, we are able to prove the existence result of adjoints of quasi-functors:

Theorem 3.5.9. *Let* $\mathbf{A}, \mathbf{B}$ *be dg-categories. Assume that* $\mathbf{A}$ *is triangulated and smooth, and that* $\mathbf{B}$ *is locally perfect. Let* $T\colon \mathbf{A} \to \mathbf{B}$ *be a quasi-functor. Then,* T *admits both a left and a right adjoint.*

Proof. By hypothesis, T_A is quasi-representable, in particular perfect, for all $A \in \mathbf{A}$. So, by Lemma 3.5.8, T is a perfect bimodule. Since $\mathbf{B}$ is locally perfect, then T^B is a perfect left $\mathbf{A}$-module for all $B \in \mathbf{B}$. But $\mathbf{A}$ is triangulated, so we conclude that T^B is quasi-representable for all $B \in \mathbf{B}$, hence we conclude that T has a left adjoint, by Proposition 3.5.5.

To prove the existence of the right adjoint, we apply a similar argument to $\mathbb{L}L(T)$. Since T is right quasi-representable, then $\mathbb{L}L(T)$ is left quasi-representable, that is,

$\mathbb{L}L(T)^A$ is quasi-representable, in particular perfect, for all $A \in \mathbf{A}$. Since $\mathbf{A}$ is smooth, we have that $\mathbb{L}L(T)$ is a perfect bimodule; since $\mathbf{B}$ is locally perfect, we deduce that $\mathbb{L}L(T)_B$ is a perfect right $\mathbf{A}$-module for all $B \in \mathbf{B}$. Since $\mathbf{A}$ is triangulated, $\mathbb{L}L(T)_B$ is quasi-representable for all $B \in \mathbf{B}$, so in the end $\mathbb{L}L(T)$ is both left and right quasi-representable, and by Proposition 3.5.5 we conclude that T has a right adjoint. □

Remark 3.5.10. The above result is mentioned in [TV15], under stronger assumptions on the dg-categories, namely, saturatedness: see [TV07, Definition 2.4]. If a dg-category $\mathbf{A}$ is saturated, then in particular it is triangulated and $H^0(\mathbf{A})$ is saturated as a triangulated category ([TV08, Appendix A]), that is, any covariant or contravariant cohomological functor $H^0(\mathbf{A}) \to \mathsf{Mod}(\mathbf{k})$ of finite type is representable. It is an easy exercise to show that exact functors between saturated (and Ext-finite) triangulated categories admit adjoints: Theorem 3.5.9 can hence be viewed as an enhancement of this result in the dg framework.

3.6 The abstract dg-lift problem

As we know, any bimodule $T \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A},\mathbf{B})$ induces a $H^0(\mathbf{A})$-$H^0(\mathbf{B})$-bimodule, obtained by taking degree 0 cohomology: $H^0(T)_A^B = H^0(T_A^B)$. This mapping, analogously to (3.1.1), is functorial in the following sense:

$$H^0\colon \mathsf{K}(\mathbf{A},\mathbf{B}) \to \mathsf{Mod}(H^0(\mathbf{A}), H^0(\mathbf{B})), \tag{3.6.1}$$

where we set $\mathsf{Mod}(H^0(\mathbf{A}), H^0(\mathbf{B})) = \mathsf{Mod}(H^0(\mathbf{B}) \otimes H^0(\mathbf{A})^{\mathrm{op}})$, the category of $H^0(\mathbf{A})$-$H^0(\mathbf{B})$-bimodules. If $T \in \mathrm{qrep}^r(\mathbf{A},\mathbf{B})$ is a quasi-functor (for simplicity, assume that $\mathbf{A}$ and $\mathbf{B}$ are identified with their h-projective resolutions), then $H^0(T)_A = H^0(T_A) \cong H^0(h_{F(A)})$ is the right $H^0(\mathbf{B})$-module represented by $F(A)$; hence, $H^0(T)$ is right representable as a $H^0(\mathbf{A})$-$H^0(\mathbf{B})$-bimodule, and in particular it yields a functor $H^0(\mathbf{A}) \to H^0(\mathbf{B})$, which we denote by $H^0(T)$, abusing notation. The mapping $T \mapsto H^0(T)$ is easily seen to be functorial, and in the end we get a functor

$$\Phi^{\mathbf{A}\to\mathbf{B}}\colon H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A},\mathbf{B})) \to \mathrm{Fun}(H^0(\mathbf{A}), H^0(\mathbf{B})), \tag{3.6.2}$$

which is actually a revisitation of the H^0 functor (3.6.1). En passant, it is worth mentioning that this functor preserves adjunctions:

Lemma 3.6.1. *An adjunction $T \dashv S\colon \mathbf{A} \leftrightarrows \mathbf{B}$ of quasi-functors induces an adjunction of functors $H^0(T) \dashv H^0(S)\colon H^0(\mathbf{A}) \leftrightarrows H^0(\mathbf{B})$.*

Proof. By hypothesis we have:

$$\begin{aligned} H^0(T)_A &\cong H^0(\mathbf{B})(-, F(A)), \\ H^0(S)_B &\cong H^0(\mathbf{A})(-, G(B)), \end{aligned}$$

and there is a unique way to define functors $F\colon H^0(\mathbf{A}) \to H^0(\mathbf{B})$ and $G\colon \mathbf{H}^0(B) \to \mathbf{H}^0(A)$ such that the above isomorphisms are natural respectively in A and B. By definition, $H^0(T)$ is identified with F and $H^0(S)$ is identified with G. We know (Proposition 3.5.5) that $S \overset{\text{qis}}{\approx} \mathbb{L}L(T)$. In particular, we find that

$$H^0(S)^A \cong H^0(\mathbf{B})(F(A), -),$$

naturally in A. We conclude that

$$H^0(S) \cong H^0(\mathbf{B})(F(-), -) \cong H^0(\mathbf{A})(-, G(-)),$$

as required. □

The *dg-lift uniqueness problem*, which will be addressed in the following part of the work, amounts to understanding in which cases $\Phi^{\mathbf{A}\to\mathbf{B}}$ is essentially injective, that is: given quasi-functors T_1, T_2 such that $H^0(T_1) \cong H^0(T_2)$, is it true that $T_1 \overset{\text{qis}}{\approx} T_2$?

Remark 3.6.2. If $\mathbf{A}$ and $\mathbf{B}$ are pretriangulated, then a quasi-functor $T\colon \mathbf{A} \to \mathbf{B}$ yields an *exact* functor $H^0(T)\colon H^0(\mathbf{A}) \to H^0(\mathbf{B})$. In this case, we will always view $\Phi^{\mathbf{A}\to\mathbf{B}}$ as taking values in $\mathrm{Fun}_{\mathrm{ex}}(H^0(\mathbf{A}, H^0(\mathbf{B}))$:

$$\Phi^{\mathbf{A}\to\mathbf{B}}\colon H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B})) \to \mathrm{Fun}_{\mathrm{ex}}(H^0(\mathbf{A}), H^0(\mathbf{B})).$$

The dg-lift uniquness problem, stated in its generality, is difficult; still, we are able to obtain a general "duality result":

Proposition 3.6.3. *Let $\mathbf{A}, \mathbf{B}$ be dg-categories, and assume that every quasi-functor $\mathbf{A} \to \mathbf{B}$ admits a left or a right adjoint (for example, assume the hypotheses of Theorem 3.5.9). Then, if $\Phi^{\mathbf{B}\to\mathbf{A}}$ is essentially injective, so is $\Phi^{\mathbf{A}\to\mathbf{B}}$.*

Proof. Assume $F, F'\colon \mathbf{A} \to \mathbf{B}$ are quasi-functors such that $H^0(F) \cong H^0(F')$. By hypothesis, F and F' have (right or left) adjoints G and G'. By Lemma 3.6.1, $H^0(G)$ and $H^0(G')$ are adjoints of $H^0(F)$ and $H^0(F')$, so they have to be isomorphic: $H^0(G) \cong H^0(G')$. By hypothesis, $G \overset{\text{qis}}{\approx} G'$. So, by the uniqueness of adjoints up to isomorphism, we conclude that $F \overset{\text{qis}}{\approx} F'$, as required. □

Also, it is worth to address the trivial case:

Lemma 3.6.4. *View $\mathbf{k}$ as a dg-category, and let $\mathbf{B}$ be a dg-category. Then, $\Phi^{\mathbf{k}\to\mathbf{B}}$ is an equivalence.*

Proof. We identify $H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{k}, \mathbf{B}))$ to right quasi-representable bimodules $\mathbf{k} \rightsquigarrow \mathbf{B}$, that is, to the category $\mathrm{qrep}(\mathbf{B})$ of quasi-representable right $\mathbf{B}$-modules. Analogously, we identify $\mathrm{Fun}(\mathbf{k}, H^0(\mathbf{B}))$ to the category $H^0(\mathbf{B})$. The diagram

$$\begin{array}{ccc} \mathrm{qrep}(\mathbf{B}) & \xrightarrow{\Phi^{\mathbf{k}\to\mathbf{B}}} & H^0(\mathbf{B}) \\ {\scriptstyle\sim}\uparrow & \nearrow\!\!\!\!= & \\ H^0(\mathbf{B}) & & \end{array}$$

is commutative, where the vertical arrow is induced by the derived Yoneda embedding. Our claim follows. □

In many situations, we will be studying dg-functors whose domain dg-category **A** is (pre)triangulated and generated by a simpler dg-category: for instance, $\mathbf{A} \overset{\text{qe}}{\approx} \operatorname{per}_{\text{dg}}(\mathbf{C})$. In this case, the dg-lift uniquness problem can be reduced to generators:

Lemma 3.6.5. *Let* **A** *and* **B** *be triangulated dg-categories, and assume that* $\mathbf{A} \overset{\text{qe}}{\approx} \operatorname{per}_{\text{dg}}(\mathbf{C})$ *for some dg-category* **C**. *Then,* $\Phi^{\mathbf{A}\to\mathbf{B}}$ *is essentially injective if* $\Phi^{\mathbf{C}\to\mathbf{B}}$ *is such.*

Proof. Without loss of generality, we may identify $\mathbf{A} = \operatorname{per}_{\text{dg}}(\mathbf{C})$. There is a commutative diagram:

$$\begin{array}{ccc} H^0(\mathbb{R}\underline{\operatorname{Hom}}(\operatorname{per}_{\text{dg}}(\mathbf{C}),\mathbf{B})) & \xrightarrow{\Phi^{\mathbf{A}\to\mathbf{B}}} & \operatorname{Fun}_{\text{ex}}(H^0(\operatorname{per}_{\text{dg}}(\mathbf{C})), H^0(\mathbf{B})) \\ \downarrow{\scriptstyle\sim} & & \downarrow \\ H^0(\mathbb{R}\underline{\operatorname{Hom}}(\mathbf{C},\mathbf{B})) & \xrightarrow{\Phi^{\mathbf{C}\to\mathbf{B}}} & \operatorname{Fun}(H^0(\mathbf{C}), H^0(\mathbf{B})), \end{array}$$

where the left vertical arrow is induced by the Yoneda embedding $\mathbf{C} \hookrightarrow \operatorname{per}_{\text{dg}}(\mathbf{C})$, and the right vertical arrow is induced by its zeroth cohomology: $H^0(\mathbf{C}) \hookrightarrow H^0(\operatorname{per}_{\text{dg}}(\mathbf{C}))$. By Proposition 3.5.4, the left vertical arrow is an isomorphism; the claim now follows from a direct argument. □

Remark 3.6.6. The above argument works also if we consider **A** and **B** pretriangulated, and $\mathbf{A} \overset{\text{qe}}{\approx} \operatorname{pretr}(\mathbf{C})$.

To end the chapter, we prove another relevant property of the functor (3.6.2):

Proposition 3.6.7. *The functor* (3.6.2) *reflects isomorphisms.*

Proof. Assume, for simplicity, that **A** and **B** are h-projective. Recall that a morphism $T \to T'$ in $H^0(\mathbb{R}\underline{\operatorname{Hom}}(\mathbf{A},\mathbf{B})) = \operatorname{qrep}^r(\mathbf{A},\mathbf{B})$ is given by a roof

$$T \xleftarrow{\approx} Q(T) \to T'$$

in $\mathsf{K}(\mathbf{A},\mathbf{B})$, where the arrow $Q(T) \to T$ is a quasi-isomorphism. So, it sufficient to prove that any morphism of bimodules $\varphi : T \to T'$ is a quasi-isomorphism if $H^0(\varphi) : H^0(T) \to H^0(T')$ is an isomorphism. Now, φ is a quasi-isomorphism if and only if $\varphi_A : T_A \to T'_A$ is an isomorphism in $\mathsf{D}(\mathbf{B})$ for all $A \in \mathbf{A}$, which is equivalent to requiring that $\varphi'_A : \mathbf{B}(-,F(A)) \to \mathbf{B}(-,F'(A))$ is an isomorphism in $\mathsf{D}(\mathbf{B})$, where φ'_A is the unique morphism in $\mathsf{D}(\mathbf{B})$ such that the following diagram is commutative in $\mathsf{D}(\mathbf{B})$:

$$\begin{array}{ccc} T_A & \xrightarrow{\varphi_A} & T'_A \\ \downarrow{\scriptstyle\approx} & & \downarrow{\scriptstyle\approx} \\ \mathbf{B}(-,F(A)) & \xrightarrow{\varphi'_A} & \mathbf{B}(-,F'(A)). \end{array}$$

Now, by the derived Yoneda embedding of $\mathbf{B}$, φ'_A is a quasi-isomorphism if and only if $\varphi'_A(1_{F(A)}) \in \mathbf{B}(F(A), F'(A))$ is a homotopy equivalence. This means that $[\varphi'_A(1_{F(A)})] = H^0(\varphi'_A)([1_{F(A)}])$ is an isomorphism in $H^0(\mathbf{B})$, so by the Yoneda embedding of the (ordinary) category $H^0(\mathbf{B})$ this is equivalent to requiring that

$$H^0(\varphi'_A)\colon H^0(\mathbf{B})(-, F(A)) \to H^0(\mathbf{B})(-, F'(A))$$

is an isomorphism in $\mathsf{Mod}(H^0(\mathbf{B}))$. Taking H^0, the above commutative diagram becomes:

$$\begin{array}{ccc} H^0(T)_A & \xrightarrow{H^0(\varphi)_A} & H^0(T')_A \\ \downarrow \sim & & \downarrow \sim \\ \mathbf{H}^0(B)(-, F(A)) & \xrightarrow{H^0(\varphi'_A)} & \mathbf{H}^0(B)(-, F'(A)). \end{array}$$

By hypothesis, $H^0(\varphi)_A$ is an isomorphism, so we deduce that $H^0(\varphi'_A)$ is an isomorphism for all $A \in \mathbf{A}$; by the above discussion, this implies that $\varphi_A \colon T_A \to T'_A$ is a quasi-isomorphism for all A, and we are done. $\square$

Part II

Uniqueness results of dg-lifts and applications

Chapter 4

Exceptional sequences and glueings

The dg-lift uniqueness problem becomes a little simpler if we put additional hypotheses on the domain dg-category $\mathbf{A}$. Namely, we are interested in the case where $\mathbf{A}$ is triangulated and $H^0(\mathbf{A})$ has a *strong and full exceptional sequence*, which is a very particular case of *semiorthogonal decomposition.* Such categories arise in the geometric setting, and they constitute, in some sense, the simplest interesting framework for our problem.

4.1 Semiorthogonal decompositions

We start by recalling the definitions and the main results about exceptional sequences and semiorthogonal decompositions in triangulated categories. Possible references for this part are [BLL04] and [KL14].

Definition 4.1.1. Let $\mathbf{T}$ be a triangulated category. A *semiorthogonal decomposition* in two components of $\mathbf{T}$ consists of two strictly full triangulated subcategories $\mathbf{T}_1$ and $\mathbf{T}_2$ of $\mathbf{T}$ (with embedding functors $i_j : \mathbf{T}_j \hookrightarrow \mathbf{T}$ for $j = 1, 2$), such that:

- $\mathbf{T}(i_2(A_2), i_1(A_1)) \cong 0$ for all $A_1 \in \mathbf{T}_1$ and $A_2 \in \mathbf{T}_2$.
- For any $A \in \mathbf{T}$ there is a distinguished triangle
$$i_2(A_2) \to A \to i_1(A_1) \to i_2(A_2)[1], \tag{4.1.1}$$
with $A_i \in \mathbf{T}_i$ for $i = 1, 2$.

If $\mathbf{T}$ is the semiorthogonal decomposition of $\mathbf{T}_1$ and $\mathbf{T}_2$, we write
$$\mathbf{T} = \langle \mathbf{T}_1, \mathbf{T}_2 \rangle.$$

Remark 4.1.2. The factor $\mathbf{T}_1$ of $\langle \mathbf{T}_1, \mathbf{T}_2 \rangle$ is the *right orthogonal* $\mathbf{T}_2^\perp$ of $\mathbf{T}_2$:
$$\mathbf{T}_2^\perp = \{A \in \mathbf{T} : \mathbf{T}(i_2(A_2), A) \cong 0 \quad \forall\, A_2 \in \mathbf{T}_2\}.$$

Analogously, $\mathbf{T}_2$ is the *left orthogonal* ${}^{\perp}\mathbf{T}_1$ of $\mathbf{T}_1$. From this, we can easily show that $\mathbf{T}_1$ and $\mathbf{T}_2$ are *thick* triangulated subcategories of $\mathbf{T}$ (that is, closed under direct summands).

We may define semiorthogonal decompositions with more than two terms, inductively. For example, a three-term semiorthogonal decomposition of $\mathbf{T} = \langle \mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3 \rangle$ consists of three strictly full subcategories $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$ such that there are two-term semiorthogonal decompositions $\mathbf{T} = \langle \mathbf{T}_1, \mathbf{T}_2 \cup \mathbf{T}_3 \rangle$ and $\mathbf{T}_2 \cup \mathbf{T}_3 = \langle \mathbf{T}_2, \mathbf{T}_3 \rangle$ (or, equivalently, $\mathbf{T} = \langle \mathbf{T}_1 \cup \mathbf{T}_2, \mathbf{T}_3 \rangle$ and $\mathbf{T}_1 \cup \mathbf{T}_2 = \langle \mathbf{T}_1, \mathbf{T}_2 \rangle$); here we have abused notation, writing $\mathbf{T}_1 \cup \mathbf{T}_2$ for the triangulated subcategory of $\mathbf{T}$ generated by that set of objects. Exceptional sequences are particular examples of semiorthogonal decompositions:

Definition 4.1.3. Let $\mathbf{T}$ be a triangulated category. An object $E \in \mathbf{T}$ is called *exceptional* if

$$\mathbf{T}(E, E[h]) = \begin{cases} \mathbf{k} & \text{if } h = 0 \\ 0 & \text{if } h \neq 0. \end{cases}$$

An *exceptional sequence* is a sequence of exceptional objects $(E_1, \dots, E_n)$ in $\mathbf{T}$ such that $\mathbf{T}(E_i, E_j[h]) = 0$ for all $i > j$ and all $h \in \mathbb{Z}$. The exceptional sequence is called *full* if $\{E_1, \dots, E_n\}$ is a set of generators of the triangulated category $\mathbf{T}$. In this case, we write

$$\mathbf{T} = \langle E_1, \dots E_n \rangle.$$

Moreover, $(E_1, \dots, E_n)$ is said to be *strong* if $\mathbf{T}(E_i, E_j[h]) = 0$ for all $i, j = 1, \dots, n$ and all $h \in \mathbb{Z} \setminus \{0\}$.

The notation $\mathbf{T} = \langle E_1, \dots, E_n \rangle$ is consistent with the fact that a full exceptional sequence gives a semiorthogonal decomposition of $\mathbf{T}$, obtained with the triangulated subcategories generated by each E_i.

Next, we establish some basic results about semiorthogonal decompositions. We assume $\mathbf{T} = \langle \mathbf{T}_1, \mathbf{T}_2 \rangle$. The following lemma ensures that we can, in fact, speak of *the* distinguished triangle (4.1.1) associated to an object $A \in \mathbf{T}$:

Lemma 4.1.4. *Let $A, B \in \mathbf{T} = \langle \mathbf{T}_1, \mathbf{T}_2 \rangle$, and let*

$$T_A = (i_2(A_2) \to A \to i_1(A_1) \to i_2(A_2)[1]),$$
$$T_B = (i_2(B_2) \to B \to i_1(B_1) \to i_2(B_2)[1])$$

be two distinguished triangles associated to A and B, given by the definition. Moreover, let $f\colon A \to B$ be a morphism in $\mathbf{T}$. Then, there exists a unique morphism $T_A \to T_B$ of distinguished triangles extending f.

Proof. Apply the cohomological functor $\mathbf{T}(-, i_1(B_1))$ to T_A and $\mathbf{T}(i_2(A_2), -)$ to T_B. The semiorthogonality hypothesis gives isomorphisms:

$$\mathbf{T}(i_1(A_1), i_1(B_1)) \xrightarrow{\sim} \mathbf{T}(A, i_1(B_1)),$$
$$\mathbf{T}(i_2(A_2), i_2(B_2)) \xrightarrow{\sim} \mathbf{T}(i_2(A_2), B).$$

Then, there exist unique maps $u\colon A_1 \to B_1$ and $v\colon A_2 \to B_2$ such that the following diagram is commutative:

$$\begin{array}{ccccc} i_2(A_2) & \longrightarrow & A & \longrightarrow & i_1(A_1) \\ \downarrow{\scriptstyle i_2(v)} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle i_1(u)} \\ i_2(B_2) & \longrightarrow & B & \longrightarrow & i_1(B_1). \end{array}$$

Applying TR3, we find that those maps actually define a morphism of distinguished triangles:

$$\begin{array}{ccccccc} i_2(A_2) & \longrightarrow & A & \longrightarrow & i_1(A_1) & \longrightarrow & i_2(A_2)[1] \\ \downarrow{\scriptstyle i_2(v)} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle i_1(u)} & & \downarrow{\scriptstyle i_2(v)[1]} \\ i_2(B_2) & \longrightarrow & B & \longrightarrow & i_1(B_1) & \longrightarrow & i_2(B_2)[1]. \end{array}$$

This morphism is uniquely determined by f. □

Corollary 4.1.5. *In the above setting, the embedding functor $i_1 : \mathbf{T}_1 \hookrightarrow \mathbf{T}$ has a left adjoint $i_1^* : \mathbf{T} \to \mathbf{T}_1$, and the embedding functor $i_2 : \mathbf{T}_2 \hookrightarrow \mathbf{T}$ has a right adjoint $i_2^! : \mathbf{T} \to \mathbf{T}_2$. Given $A \in \mathbf{T}$, its associated distinguished triangle* (4.1.1) *is*

$$i_2(i_2^!(A)) \to A \to i_1(i_1^*(A)) \to i_2(i_2^!(A))[1], \tag{4.1.2}$$

and the morphism $A \to i_1(i_1^*(A))$ (resp. $i_2(i_2^!(A)) \to A$) is a component of the unit (resp. counit) of the adjunction $i_1^* \dashv i_1$ (resp. $i_2 \dashv i_2^!$).

Proof. Let $A \in \mathbf{T}$. Define $i_1^*(A)$ and $i_2^!(A)$ as the objects in $\mathbf{T}_1$ and $\mathbf{T}_2$ in the distinguished triangle

$$T_A = (i_2(i_2^!(A)) \to A \to i_1(i_1^*(A)) \to i_2(i_2^!(A))[1])$$

given by the definition. Next, we show that the maps $A \to i_1(i_1^*(A))$ and $i_2(i_2^!(A)) \to A$ above satisfy the required universal properties, thus giving the functors i_1^* and $i_2^!$ together with the unit and the counit of the adjunctions. Let $A_1 \in \mathbf{T}_1$ and let $A \to i_1(A_1)$ be a morphism. We associate to $i_1(A_1)$ the following distinguished triangle:

$$T_{i_1(A)} = (\ 0 \longrightarrow i_1(A_1) = i_1(A_1) \longrightarrow 0\ );$$

then, by Lemma 4.1.4, the map $A \to i_1(A_1)$ induces a unique morphism $T_A \to T_{i_1(A_1)}$. In particular, there is a unique map $i_1^*(A) \to A_1$ such that the diagram

$$\begin{array}{ccc} A & \longrightarrow & i_1(i_1^*(A)) \\ \downarrow & & \downarrow \\ i_1(A_1) & = & i_1(A_1). \end{array}$$

is commutative. This is the required universal property. A similar argument gives the right adjoint $i_2^!$. □

In some sense, an object $A \in \mathbf{T} = \langle \mathbf{T}_1, \mathbf{T}_2 \rangle$ is obtained as a "glueing" of $i_1^*(A)$ and $i_2^!(A)$, along the map $i_1(i_1^*(A)) \to i_2(i_2^!(A))[1]$. Let us make this intuition more precise.

Definition 4.1.6. Let $\mathbf{C}, \mathbf{D}$ be ($\mathbf{k}$-linear) categories, and let $N \in \mathsf{Mod}(\mathbf{D}, \mathbf{C})$ be a bimodule, covariant in $\mathbf{D}$ and contravariant in $\mathbf{C}$. The *glueing* of $\mathbf{C}$ and $\mathbf{D}$ along N is the category $\mathbf{C} \times_N \mathbf{D}$, defined as follows. Objects are triples (C, D, m), where $C \in \mathbf{C}$, $D \in \mathbf{D}$ and $m \in N(C, D) = N_D^C$. Morphisms $(C, D, m) \to (C', D', m')$ are couples (u, v), where $u \in \mathbf{C}(C, C')$, $v \in \mathbf{D}(D, D')$, such that $m'u = vm$. Compositions and identities are defined termwise.

Remark 4.1.7. It is sometimes useful to interpret an element $m \in N(C, D)$ as a "generalised morphism" from C to D, which will be stylised as $m \colon C \rightharpoonup D$. Hence, morphisms in $\mathbf{C} \times_N \mathbf{D}$ can be interpreted as commutative diagrams such as

$$\begin{array}{ccc} C & \xrightharpoonup{m} & D \\ \downarrow u & & \downarrow v \\ C' & \xrightharpoonup{m'} & D'. \end{array}$$

Let $\mathbf{T} = \langle \mathbf{T}_1, \mathbf{T}_2 \rangle$ be a semiorthogonal decomposition. We define the *glueing bimodule* $\Phi \in \mathsf{Mod}(\mathbf{T}_2, \mathbf{T}_1)$ as

$$\Phi(A_1, A_2) = \mathbf{T}(i_1(A_1), i_2(A_2)[1]) \tag{4.1.3}$$

with the obvious actions. There is a natural functor

$$\mathbf{T} \longrightarrow \mathbf{T}_1 \times_\Phi \mathbf{T}_2, \tag{4.1.4}$$

which is defined as follows. Given $A \in \mathbf{T}$, let

$$i_2(i_2^!(A)) \to A \to i_1(i_1^*(A)) \xrightarrow{m_A} i_2(i_2^!(A))[1]$$

be its associated distinguished triangle (determined up to isomorphism). We map A to the object $(i_1^*(A), i_2^!(A), m_A) \in \mathbf{T}_1 \times_\Phi \mathbf{T}_2$. Then, let $f : A \to B$ be a morphism in $\mathbf{T}$. By Proposition 4.1.4 and Corollary 4.1.5, we have a morphism of distinguished triangles:

$$\begin{array}{ccccccc} i_2(i_2^!(A)) & \longrightarrow & A & \longrightarrow & i_1(i_1^*(A)) & \xrightarrow{m_A} & i_2(i_2^!(A))[1] \\ \downarrow{\scriptstyle i_2(i_2^!(f))} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle i_1(i_1^*(f))} & & \downarrow{\scriptstyle i_2(i_2^!(f))[1]} \\ i_2(i_2^!(B)) & \longrightarrow & B & \longrightarrow & i_1(i_1^*(B)) & \xrightarrow{m_B} & i_2(i_2^!(B))[1]. \end{array} \tag{4.1.5}$$

So, we map f to the couple $(i_1^*(f), i_2^!(f))$. Notice that the definition of this functor depends on the choice of the above maps m_A, but different choices give isomorphic functors. We have the following result, which is essentially a more precise version of [KL14, Lemma 2.5], and explains how objects in $\mathbf{T}$ can be thought as "glueings":

Proposition 4.1.8. *The functor* (4.1.4) *is essentially surjective, full, and reflects isomorphisms. In particular, it is essentially injective.*

Proof. Let $(A_1, A_2, m) \in \mathbf{T}_1 \times_\Phi \mathbf{T}_2$. Take $A = \mathrm{C}(m)[-1]$, obtaining the distinguished triangle:

$$A_2 \to A \to A_1 \xrightarrow{m} A_2[1],$$

which is isomorphic to

$$i_2(i_2^!(A)) \to A \to i_1(i_1^*(A)) \xrightarrow{m_A} i_2(i_2^!(A))[1]$$

by Lemma 4.1.4. So, the functor maps A to $(i_1^*(A), i_2^!(A), m_A)$, which is isomorphic to (A_1, A_2, m). This proves essential surjectivity.

Next, we prove fullness. Let $(u, v)\colon (i_1^*(A), i_2^!(A), m_A) \to (i_1^*(B), i_2^!(B), m_B)$ be a morphism. This is actually the following commutative square:

$$\begin{array}{ccc} i_1(i_1^*(A)) & \xrightarrow{m_A} & i_2(i_2^!(A))[1] \\ \downarrow{\scriptstyle i_1(u)} & & \downarrow{\scriptstyle i_2(v)[1]} \\ i_1(i_1^*(B)) & \xrightarrow{m_B} & i_2(i_2^!(B))[1]. \end{array}$$

Applying the axiom TR3, we find $f\colon A \to B$ and a morphism of distinguished triangles:

$$\begin{array}{ccccccc} i_2(i_2^!(A)) & \longrightarrow & A & \longrightarrow & i_1(i_1^*(A)) & \xrightarrow{m_A} & i_2(i_2^!(A))[1] \\ \downarrow{\scriptstyle i_2(v)} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle i_1(u)} & & \downarrow{\scriptstyle i_2(v)[1]} \\ i_2(i_2^!(B)) & \longrightarrow & B & \longrightarrow & i_1(i_1^*(B)) & \xrightarrow{m_B} & i_2(i_2^!(B))[1]. \end{array}$$

By Lemma 4.1.4, this morphism is uniquely induced by f, in particular $u = i_1^*(f)$ and $v = i_2^!(f)$. This shows fullness.

Finally, it is easy to show that the functor reflects isomorphisms: if $(i_1^*(f), i_2^!(f))$ is an isomorphism, then f is too an isomorphism, since it is part of the morphism of distinguished triangles (4.1.5), where the two other components are assumed to be isomorphisms. □

In some cases, the glueing bimodule Φ of a semiorthogonal decomposition $\mathbf{T} = \langle \mathbf{T}_1, \mathbf{T}_2 \rangle$ happens to be right representable, hence inducing what we call the *glueing functor* $\varphi\colon \mathbf{T}_2 \to \mathbf{T}_1$ (determined up to isomorphism):

$$\mathbf{T}(i_1(A_1), i_2(A_2)[1]) \cong \mathbf{T}_1(A_1, \varphi(A_2)), \tag{4.1.6}$$

naturally in $A_1 \in \mathbf{T}_1$.

Proposition 4.1.9. *Let $\mathbf{T} = \langle \mathbf{T}_1, \mathbf{T}_2 \rangle$ be a semiorthogonal decomposition. Then, the existence of a glueing functor $\varphi\colon \mathbf{T}_2 \to \mathbf{T}_1$ is equivalent to the existence of a right adjoint $i_1^!$ of the inclusion $i_1\colon \mathbf{T}_1 \to \mathbf{T}$.*

Proof. Assume that the right adjoint $i_1^!$ exists. Then, we have a natural isomorphism

$$\mathbf{T}(i_1(A_1), i_2(A_2)[1]) \cong \mathbf{T}_1(A_1, i_1^! i_2(A_2)[1]),$$

and so $\varphi = i_1^! i_2[1]$ is a glueing functor.

Conversely, assume the existence of $\varphi\colon \mathbf{T}_2 \to \mathbf{T}_1$, together with a natural isomorphism

$$\mathbf{T}(i_1(A_1), i_2(A_2)[1]) \cong \mathbf{T}_1(A_1, \varphi(A_2)).$$

Denote by $\gamma_{A_2}\colon i_1\varphi(A_2) \to i_2(A_2)[1]$ the counit, so that for any $f\colon i_1(A_1) \to i_2(A_2)[1]$ there exists a unique $f'\colon A_1 \to \varphi(A_2)$ such that $f = \gamma_{A_2} i_1(f')$. Now, let $X \in \mathbf{T}$. Its canonical distinguished triangle gives a morphism $i_1 i_1^*(X) \to i_2 i_2^!(X)[1]$, and moreover there exists a unique $f_X : i_1^*(X) \to \varphi i_2^!(X)$ such that the following diagram is commutative:

$$\begin{array}{ccc} i_1 i_1^*(X) & \xrightarrow{i_1(f_X)} & i_1\varphi i_2^!(X) \\ \| & & \downarrow{\scriptstyle \gamma_{i_2^!(X)}} \\ i_1 i_1^*(X) & \longrightarrow & i_2 i_2^!(X)[1]. \end{array}$$

Next, we set $i_1^!(X) = \mathrm{C}(f_X)[-1]$. We may complete the above commutative square to a morphism of distinguished triangles:

$$\begin{array}{ccccccc} i_1\varphi i_2^!(X)[-1] & \longrightarrow & i_1 i_1^!(X) & \longrightarrow & i_1 i_1^*(X) & \xrightarrow{i_1(f_X)} & i_1\varphi i_2^!(X) \\ {\scriptstyle \gamma_{i_2^!(X)}[-1]}\downarrow & & \vdots{\scriptstyle \xi_X} & & \| & & \downarrow{\scriptstyle \gamma_{i_2^!(X)}} \\ i_2 i_2^!(X) & \longrightarrow & X & \longrightarrow & i_1 i_1^*(X) & \longrightarrow & i_2 i_2^!(X)[1]. \end{array}$$

Now, $(\gamma_{i_2^!(X)})_* : \mathbf{T}(i_1(-), i_1\varphi i_2^!(X)) \to \mathbf{T}(i_1(-), i_2 i_2^!(X)[1])$ is an isomorphism (it is essentially the natural isomorphism which defines φ), and so by the five lemma $(\xi_X)_* : \mathbf{T}(i_1(-), i_1 i_1^!(X)) \to \mathbf{T}(i_1(-), X)$ is also an isomorphism. Since i_1 is fully faithful, we obtain an isomorphism

$$\mathbf{T}(i_1(-), X) \cong \mathbf{T}_1(-, i_1^!(X)),$$

and we are done, recalling that we may define $i_1^!$ on morphisms in a unique way which makes the above isomorphism natural in X. □

4.2 The dg-glueing construction

The notion of semiorthogonal decomposition has been enhanced to the dg framework: this is achieved with the *glueing* of two dg-categories along a dg-bimodule, which is a "homotopy coherent" incarnation of the glueing of Definition 4.1.6. The definition and the main properties of this construction, except Propositions 4.2.2 and 4.3.12, are all taken from [KL14], sometimes with some slight modifications.

Definition 4.2.1. Let $\mathbf{A}$ and $\mathbf{B}$ be dg-categories, and let $N\colon \mathbf{B} \rightsquigarrow \mathbf{A}$ be a dg-bimodule. The (differential graded, homotopy coherent) *glueing* of $\mathbf{A}$ and $\mathbf{B}$ along N is the dg-category $\mathbf{A} \times_N \mathbf{B}$, defined in the following way. Objects are triples (A, B, m), where $A \in \mathbf{A}$, $B \in \mathbf{B}$, and $m \in Z^0(N(A,B))$. A degree n morphism $(A,B,m) \to (A',B',m')$ is given by a lower triangular matrix

$$(u,v,h) = \begin{pmatrix} u & 0 \\ h & v \end{pmatrix},$$

where $u \in \mathbf{A}(A,A')^n$, $v \in \mathbf{B}(B,B')^n$ and $h \in N(A,B')^{n-1}$. Compositions are defined by matrix multiplication with a sign rule:

$$\begin{pmatrix} u' & 0 \\ h' & v' \end{pmatrix}\begin{pmatrix} u & 0 \\ h & v \end{pmatrix} = \begin{pmatrix} u'u & 0 \\ (-1)^n h'u + v'h & v'v \end{pmatrix},$$

whenever (u,v,h) has degree n. The differential of a morphism $(u,v,h)\colon (A,B,m) \to (A',B',m')$ of degree n is defined by

$$d\begin{pmatrix} u & 0 \\ h & v \end{pmatrix} = \begin{pmatrix} du & 0 \\ dh + (-1)^n(m'u - vm) & dv \end{pmatrix}.$$

As we may expect, the dg-glueing induces an ordinary glueing in cohomology. Let $\mathbf{A}, \mathbf{B}$ be dg-categories. Recall that, given $N \in \mathsf{C}_{\mathrm{dg}}(\mathbf{B}, \mathbf{A})$, we can take its cohomology and obtain $H^0(N) \in \mathsf{Mod}(H^0(\mathbf{B}), H^0(\mathbf{A}))$. Hence, we can form the (ordinary) glueing $H^0(\mathbf{A}) \times_{H^0(N)} H^0(\mathbf{B})$. There is a natural functor:

$$\begin{aligned} H^0(\mathbf{A} \times_N \mathbf{B}) &\longrightarrow H^0(\mathbf{A}) \times_{H^0(N)} H^0(\mathbf{B}), \\ (A,B,m) &\mapsto (A,B,[m]), \\ [(u,v,h)] &\mapsto ([u],[v]). \end{aligned} \tag{4.2.1}$$

We get the following result, which is conceptually analogue to Proposition 4.1.8.

Proposition 4.2.2. *The above functor is strictly surjective on objects, full and reflects isomorphisms.*

Proof. Surjectivity on objects is clear. Fullness is simple: if $([u],[v])\colon (A,B,[m]) \to (A',B',[m'])$, then $[vm] = [m'u]$ as elements of $H^0(N(A,B'))$, that is, there exists $h \in N(A,B')^{-1}$ such that $dh = vm - m'u$. Then, $([u],[v])$ is the image of $[(u,v,h)]$.

Now, we prove the last assertion. Let $[(u,v,h)]\colon (A,B,m) \to (A',B',m')$ be a morphism such that $([u],[v])$ is an isomorphism. By hypothesis we have $dh = vm - m'u$, and an inverse $([u'],[v'])$ of $([u],[v])$. That is, $u'\colon A' \to A$ and $v'\colon B' \to B$ are closed degree 0 maps such that

$$\begin{cases} u'u = 1_A + d\tilde{u}, \\ v'v = 1_B + d\tilde{v}, \end{cases}$$

$$\begin{cases} uu' = 1_{A'} + d\tilde{u'}, \\ vv' = 1_{B'} + d\tilde{v'}, \end{cases}$$

$$v'm' - mu' = dh',$$

for suitable maps $\tilde{u}, \tilde{u'}, \tilde{v}, \tilde{v'}$ of degree -1, and for a suitable $h' \in N(A', B)^{-1}$. Let us construct a left and a right inverse for $[(u, v, h)]$ in $H^0(\mathbf{A} \times_N \mathbf{B})$, namely $[(u', v', h' + z_0')]$ and $[(u', v', h' + z_1')]$, where z_0', z_1' are suitable elements of $Z^{-1}(A', B)$ to be defined (it is essential to find z_0', z_0' closed, for it ensures us that $(u', v', h' + z_0')$ and $(u', v', h' + z_1')$ are also closed). *A posteriori* the equality $[(u', v', h' + z_0')] = [(u', v', h' + z_1')]$ will hold, and the (double-sided) inverse of $[(u, v, h)]$ will be established. We start by setting

$$r = -m\tilde{u} + \tilde{v}m - h'u - v'h,$$
$$r' = -m'\tilde{u'} + \tilde{v'}m' - hu' - vh'.$$

A direct computation gives $dr = 0, dr' = 0$. We define:

$$z_0' = ru',$$
$$z_1' = v'r'.$$

z_0' and z_1' are indeed closed; we have $[z_0'u] = [r]$ and $[vz_1'] = [r']$ in $H^{-1}(N(A, B))$, so

$$(h' + z_0')u + v'h = d\tilde{h} - m\tilde{u} + \tilde{v}m,$$
$$hu' + v(h' + z_1') = d\tilde{h'} - m'\tilde{u'} + \tilde{v'}m',$$

for suitable elements $\tilde{h}, \tilde{h'} \in N(A, B)^{-2}$. Finally we obtain

$$\begin{aligned}(u', v', h' + z_0')&(u, v, h) \\ &= (u'u, v'v, (h' + z_0')u + v'h) = (1_A, 1_B, 0) + (d\tilde{u}, d\tilde{v}, d\tilde{h} - m\tilde{u} + \tilde{v}m) \\ &= (1_A, 1_B, 0) + d(\tilde{u}, \tilde{v}, \tilde{h}),\end{aligned}$$

and

$$\begin{aligned}(u, v, h)&(u', v', h' + z_1') \\ &= (uu', vv', hu' + v(h' + z_1')) = (1_{A'}, 1_{B'}, 0) + (d\tilde{u'}, d\tilde{v'}, d\tilde{h'} - m'\tilde{u'} + \tilde{v'}m') \\ &= (1_{A'}, 1_{B'}, 0) + d(\tilde{u'}, \tilde{v'}, \tilde{h'}).\end{aligned}$$

Hence the proof is completed. □

Remark 4.2.3. Let $\mathbf{A}$ be a dg-category. The glueing $\mathbf{A} \times_{h_\mathbf{A}} \mathbf{A}$ of $\mathbf{A}$ with itself along the diagonal bimodule is by definition equal to the dg-category $\underline{\mathrm{Mor}}\,\mathbf{A}$ of morphisms in $\mathbf{A}$. Hence, we see that the above functor (4.2.1) is actually a generalisation of the natural functor (1.3.6). Recalling Remark 1.3.16, we find out that it can be viewed as an incarnation of $\Phi^{\Delta^1 \to \mathbf{A}}$:

$$\Phi^{\Delta^1 \to \mathbf{A}} \colon H^0(\underline{\mathrm{Mor}}\,\mathbf{A}) = H^0(\mathbb{R}\underline{\mathrm{Hom}}(\Delta^1, \mathbf{A})) \to \mathrm{Fun}(\Delta^1, H^0(\mathbf{A})) = \mathrm{Mor}\, H^0(\mathbf{A}).$$

So, Proposition 4.2.2 actually implies the dg-lift uniqueness result when the domain dg-category is Δ^1.

As expected, the quasi-equivalence class of the glueing $\mathbf{A}_1 \times_N \mathbf{A}_2$ depends only on the quasi-isomorphism class of N and on the quasi-equivalence class of $\mathbf{A}_1$ and $\mathbf{A}_2$ (see [KL14, Proposition 4.14]. Now, assume that $\mathbf{A}_1$ and $\mathbf{A}_2$ have strict zero objects (if they are pretriangulated, this assumption can be made without loss of generality, upon replacing them with quasi-equivalent dg-categories). Then, there are natural fully faithful dg-functors:

$$\begin{aligned} i_1 \colon \mathbf{A}_1 \to \mathbf{A}_1 \times_N \mathbf{A}_2, &\quad A_1 \mapsto (A_1, 0, 0), \\ i_2 \colon \mathbf{A}_2 \to \mathbf{A}_1 \times_N \mathbf{A}_2, &\quad A_2 \mapsto (0, A_2, 0). \end{aligned}$$

i_1 has a left adjoint i_1^*, and i_2 has a right adjoint $i_2^!$, which are actually the "source" and "target" dg-functors:

$$\begin{aligned} i_1^* \colon \mathbf{A}_1 \times_N \mathbf{A}_2 \to \mathbf{A}_1, &\quad (A_1, A_2, m) \mapsto A_1, \\ i_2^! \colon \mathbf{A}_1 \times_N \mathbf{A}_2 \to \mathbf{A}_2, &\quad (A_1, A_2, m) \mapsto A_2. \end{aligned}$$

We also know that the dg-functor i_1 has a right adjoint in the bicategory Bimod of dg-bimodules. This is given by

$$i_1^! \colon \mathbf{A}_1 \times_N \mathbf{A}_2 \to \mathsf{C}_{\mathrm{dg}}(\mathbf{A}_1), \quad (A_1, A_2, m) \mapsto \mathrm{C}(m_*)[-1],$$

where $m_* \colon h_{A_1} \to N_{A_2}, f \mapsto mf$ is the morphism corresponding to $m \in N_{A_2}^{A_1}$ under the Yoneda lemma. Moreover, the following equalities are satisfied:

$$i_1^* i_1 = 1, \quad i_2^! i_2 = 1, \quad i_1^* i_2 = 0, \quad i_2^! i_1 = 0, \quad i_1^! \diamond i_2 \cong N[-1].$$

The following result tells us that the dg-glueing is actually an enhancement of the notion of semiorthogonal decomposition.

Proposition 4.2.4 ([KL14, Lemma 4.3, Corollary 4.5])**.** *Let $\mathbf{A}_1, \mathbf{A}_2$ be dg-categories, and let $N \colon \mathbf{A}_2 \rightsquigarrow \mathbf{A}_1$ be a bimodule. If $\mathbf{A}_1$ and $\mathbf{A}_2$ are pretriangulated, then so is the glueing $\mathbf{A}_1 \times_N \mathbf{A}_2$; moreover, in this case, we have a semiorthogonal decomposition*

$$H^0(\mathbf{A}_1 \times_N \mathbf{A}_2) = \langle H^0(\mathbf{A}_1), H^0(\mathbf{A}_2) \rangle \tag{4.2.2}$$

induced by the functors $H^0(i_1)$ and $H^0(i_2)$. The glueing bimodule of this semiorthogonal decomposition is isomorphic to $H^0(N)$.

This proposition has a converse statement:

Proposition 4.2.5 ([KL14, Proposition 4.10])**.** *Let $\mathbf{A}$ be a pretriangulated dg-category, and assume that there exists a semiorthogonal decomposition $H^0(\mathbf{A}) = \langle \mathbf{T}_1, \mathbf{T}_2 \rangle$. Then, letting $\mathbf{A}_i$ be the full dg-subcategory of $\mathbf{A}$ with the same objects as $\mathbf{T}_i$ ($i = 1, 2$), there is a quasi-equivalence*

$$\mathbf{A} \overset{\mathrm{qe}}{\approx} \mathbf{A}_1 \times_N \mathbf{A}_2, \tag{4.2.3}$$

where $N \colon \mathbf{A}_2 \rightsquigarrow \mathbf{A}_1$ is the bimodule obtained by the diagonal $h_{\mathbf{A}}$ by restriction and shift: $N(A_1, A_2) = \mathbf{A}(i_1(A_1), i_2(A_2)[1])$ for all $A_1 \in \mathbf{A}_1$ and $A_2 \in \mathbf{A}_2$, where i_1 and i_2 are the obvious inclusion dg-functors.

Remark 4.2.6. Actually, the proof of Proposition 4.2.5 shows that, if we identify $\mathbf{A} = \mathbf{A}_1 \times_N \mathbf{A}_2$ (up to quasi-equivalence), then the inclusion functors $i_1 \colon \mathbf{A}_1 \to \mathbf{A}$ and $i_2 \colon \mathbf{A}_2 \to \mathbf{A}$ can be identified with the canonical inclusions associated to the glueing.

Remark 4.2.7. Given a glueing $\mathbf{A}_1 \times_N \mathbf{A}_2$, it is easily seen that the right adjoint $i_1^!$ is a quasi-functor if and only if N is a quasi-functor. In this case, assuming $\mathbf{A}_1$ and $\mathbf{A}_2$ to be pretriangulated, $H^0(N)$ can be identified with the glueing functor of the semiorthogonal decomposition $\langle H^0(\mathbf{A}_1), H^0(\mathbf{A}_2)\rangle$.

Also the derived category of a glueing has a semiorthogonal decomposition with factors given by the derived categories of the factors. We will need a more precise version of this, involving bimodules $\mathbf{A}_1 \times_N \mathbf{A}_2 \to \mathbf{B}$. Set $\mathbf{A} = \mathbf{A}_1 \times_N \mathbf{A}_2$ for simplicity. First, notice that there are exact fully faithful functors

$$\begin{aligned} I_1 &= \mathbb{L}\operatorname{Ind}_{1_{\mathbf{B}^{\mathrm{op}}} \otimes i_2} \colon \mathsf{D}(\mathbf{A}_2, \mathbf{B}) \to \mathsf{D}(\mathbf{A}, \mathbf{B}), \\ I_2 &= \mathbb{L}\operatorname{Ind}_{1_{\mathbf{B}^{\mathrm{op}}} \otimes i_1} \colon \mathsf{D}(\mathbf{A}_1, \mathbf{B}) \to \mathsf{D}(\mathbf{A}, \mathbf{B}). \end{aligned}$$

As (derived) left Kan extensions of bimodules, they are left adjoints of restriction functors:

$$\begin{aligned} I_1^! &= \operatorname{Res}_{1_{\mathbf{B}^{\mathrm{op}}} \otimes i_2} = - \diamond^{\mathbb{L}} i_2, \\ I_2^! &= \operatorname{Res}_{1_{\mathbf{B}^{\mathrm{op}}} \otimes i_1} = - \diamond^{\mathbb{L}} i_1. \end{aligned}$$

Since $i_1 \dashv i_1^!$, then it can be shown that

$$- \diamond^{\mathbb{L}} i_1^! \dashv - \diamond^{\mathbb{L}} i_1 \colon \mathsf{D}(\mathbf{A}, \mathbf{B}) \leftrightarrows \mathsf{D}(\mathbf{A}_1, \mathbf{B}),$$

with counit given by composition with the counit $i_1 \diamond i_1^! \to h_{\mathbf{A}}$:

$$- \diamond^{\mathbb{L}} i_1 i_1^! \to - \diamond^{\mathbb{L}} h_{\mathbf{A}}.$$

We have the following proposition, which is adapted from [KL14, Corollary A.4]:

Proposition 4.2.8. *The above functors I_1 and I_2 induce a semiorthogonal decomposition:*

$$\mathsf{D}(\mathbf{A}, \mathbf{B}) = \langle \mathsf{D}(\mathbf{A}_2, \mathbf{B}), \mathsf{D}(\mathbf{A}_1, \mathbf{B})\rangle. \tag{4.2.4}$$

Moreover, for any $F \in \mathsf{D}(\mathbf{A}, \mathbf{B})$, there is a distinguished triangle:

$$(F \diamond^{\mathbb{L}} i_1) \diamond^{\mathbb{L}} N[-1] \to F \diamond^{\mathbb{L}} i_2 \to I_1^*(F), \tag{4.2.5}$$

where $(F \diamond^{\mathbb{L}} i_1) \diamond^{\mathbb{L}} N[-1] \to F \diamond^{\mathbb{L}} i_2$ is induced by the counit of $- \diamond^{\mathbb{L}} i_1^! \dashv - \diamond^{\mathbb{L}} i_1$:

$$(F \diamond^{\mathbb{L}} i_1) \diamond^{\mathbb{L}} N[-1] \overset{\mathrm{qis}}{\approx} F \diamond^{\mathbb{L}} (i_1 \diamond^{\mathbb{L}} i_1^!) \diamond^{\mathbb{L}} i_2 \to F \diamond^{\mathbb{L}} i_2.$$

Idea of proof. By [KL14, Lemma A.1], we have a dg-equivalence (actually, a strict dg-isomorphism):

$$(\mathbf{A}_1 \times_N \mathbf{A}_2)^{\mathrm{op}} \cong \mathbf{A}_2{}^{\mathrm{op}} \times_{N^{\mathrm{op}}} \mathbf{A}_1{}^{\mathrm{op}}.$$

Now, by [KL14, Proposition A.2], we have an equivalence of derived categories:

$$\begin{aligned}\mathsf{D}(\mathbf{A},\mathbf{B}) &= \mathsf{D}(\mathbf{B}\otimes(\mathbf{A}_2{}^{\mathrm{op}}\times_{N^{\mathrm{op}}}\mathbf{A}_1{}^{\mathrm{op}}))\\ &\cong \mathsf{D}((\mathbf{B}\otimes\mathbf{A}_2{}^{\mathrm{op}})\times_{\mathbf{B}\otimes N^{\mathrm{op}}}(\mathbf{B}\otimes\mathbf{A}_1{}^{\mathrm{op}})),\end{aligned}$$

where $\mathbf{B}\otimes N^{\mathrm{op}}$ is defined in the obvious way:

$$(\mathbf{B}\otimes N^{\mathrm{op}})((B,A_2),(B',A_1)) = \mathbf{B}(B,B')\otimes N^{\mathrm{op}}(A_2,A_1) = \mathbf{B}(B,B')\otimes N(A_1,A_2).$$

Moreover, the above equivalence is the (derived) extension of the functor which maps

$$(B,(A_2,A_1,m)) \mapsto ((B,A_2),(B,A_1),1_B\otimes m).$$

Now, the result follows from [KL14, Proposition 4.6]. □

The above semiorthogonal decomposition restricts to quasi-representable bimodules, hence giving us a characterisation of quasi-functors from a glueing ([KL14, Proposition A.7, (ii)]):

Proposition 4.2.9. *Assume that, in the framework of Proposition 4.2.8,* $\mathbf{B}$ *is pretriangulated and* $N\colon \mathbf{A}_2 \rightsquigarrow \mathbf{A}_1$ *is a quasi-functor. Then,* (4.2.4) *restricts to a semiorthogonal decomposition:*

$$\mathrm{qrep}^r(\mathbf{A},\mathbf{B}) = \langle \mathrm{qrep}^r(\mathbf{A}_2,\mathbf{B}), \mathrm{qrep}^r(\mathbf{A}_1,\mathbf{B})\rangle. \tag{4.2.6}$$

By Proposition 4.1.8, we deduce that a quasi-functor $F\colon \mathbf{A}\to\mathbf{B}$ is determined, up to isomorphism, by its components $F_1 = I_2^!(F)\colon \mathbf{A}_1\to\mathbf{B}$ and $F_2 = I_1^*(F)\colon \mathbf{A}_2\to\mathbf{B}$ (beware of indices!), and by the map $\psi\colon F_2\to F_1\diamond^{\mathbb{L}} N$, obtained by (4.2.5) as follows:

$$F_1\diamond^{\mathbb{L}} N[1]\to F\diamond^{\mathbb{L}} i_2\to F_2\xrightarrow{\psi} F_1\diamond^{\mathbb{L}} N.$$

More precisely, applying Proposition 4.1.8, we immediately get the following:

Corollary 4.2.10. *Assume the hypotheses of Proposition 4.2.9. Let* $F,G\colon \mathbf{A}\to\mathbf{B}$ *be quasi-functors, and let* $\varphi_1\colon F\diamond^{\mathbb{L}} i_1\to G\diamond^{\mathbb{L}} i_1$ *and* $\varphi_2'\colon F\diamond^{\mathbb{L}} i_2\to G\diamond^{\mathbb{L}} i_2$ *be morphisms of quasi-functors such that the diagram*

$$\begin{array}{ccc}(F\diamond^{\mathbb{L}} i_1)\diamond^{\mathbb{L}} N[-1] & \longrightarrow & F\diamond^{\mathbb{L}} i_2\\ \downarrow{\scriptstyle \varphi_1\diamond^{\mathbb{L}} N[-1]} & & \downarrow{\scriptstyle \varphi_2'}\\ (G\diamond^{\mathbb{L}} i_1)\diamond^{\mathbb{L}} N[-1] & \longrightarrow & G\diamond^{\mathbb{L}} i_2\end{array}$$

is commutative in $\mathrm{qrep}^r(\mathbf{A}_2,\mathbf{B})$*. Complete this to a morphism of distinguished triangles:*

$$\begin{array}{ccccccc}(F\diamond^{\mathbb{L}} i_1)\diamond^{\mathbb{L}} N[-1] & \longrightarrow & F\diamond^{\mathbb{L}} i_2 & \longrightarrow & I_1^*(F) & \xrightarrow{\psi_F} & (F\diamond^{\mathbb{L}} i_1)\diamond^{\mathbb{L}} N\\ \downarrow{\scriptstyle \varphi_1\diamond^{\mathbb{L}} N[-1]} & & \downarrow{\scriptstyle \varphi_2'} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_1\diamond^{\mathbb{L}} N}\\ (G\diamond^{\mathbb{L}} i_1)\diamond^{\mathbb{L}} N[-1] & \longrightarrow & G\diamond^{\mathbb{L}} i_2 & \longrightarrow & I_1^*(G) & \xrightarrow{\psi_G} & (G\diamond^{\mathbb{L}} i_1)\diamond^{\mathbb{L}} N;\end{array}$$

then, there exists a morphism $\varphi\colon F\to G$ *such that* $I_1^*(\varphi)=\varphi_2$*, and* $I_2^!(\varphi)=\varphi\diamond^{\mathbb{L}} i_1=\varphi_1$*. Moreover, if* φ_1 *and* φ_2 *are isomorphisms, so is* φ*.*

The above corollary is the main tool we will employ to address the dg-lift uniqueness problem in the case when the domain dg-category is a glueing. Namely, the attempt is to obtain the uniqueness result for the glueing assuming that it holds for the factors.

4.3 Dg-lifts and exceptional sequences

From now on, we assume that $\mathbf{k}$ is a field. With a little abuse of terminology, we say that a triangulated dg-category $\mathbf{A}$ admits a full (and strong) exceptional sequence $(E_0, \dots, E_n)$ if $\mathbf{A}$ is generated (in the sense of Definition 3.2.7) by a full dg-subcategory $\mathbb{E} = \{E_0, \dots, E_n\}$ such that $H^0(\mathbb{E})$ gives a full (and strong) exceptional sequence of $H^0(\mathbf{A})$. We state a simple preparatory result:

Lemma 4.3.1. *Let $\mathbf{A}$ be a triangulated dg-category, and assume it is generated by a full dg-subcategory $\mathbb{E}$ which has homology concentrated in degree 0. Then, $\mathbf{A}$ is quasi-equivalent to $\operatorname{per}_{\mathrm{dg}}(H^0(\mathbb{E}))$.*

Proof. By Remark 3.2.8, $\mathbf{A}$ is quasi-equivalent to $\operatorname{per}_{\mathrm{dg}}(\mathbb{E})$. By Proposition 1.3.15, $\mathbb{E}$ is quasi-equivalent to $H^0(\mathbb{E})$, hence we conclude, applying Proposition 3.2.6. □

Remark 4.3.2. We remark that any pretriangulated dg-category $\mathbf{A}$ having a full exceptional sequence is actually triangulated. More in general, if $\mathbf{A}$ is a pretriangulated dg-category such that $H^0(\mathbf{A}) = \langle H^0(\mathbf{A}_1), H^0(\mathbf{A}_2)\rangle$ and the full dg-subcategories $\mathbf{A}_1$ and $\mathbf{A}_2$ are triangulated, then so is $\mathbf{A}$. This follows from the general fact that a triangulated category is idempotent complete if it has a semiorthogonal decomposition with idempotent complete factors, see [BDF$^+$14, Lemma 4.6]. Another way to see this can be sketched as follows. First, apply Proposition 4.2.5 and view $\mathbf{A}$, up to quasi-equivalence, as a glueing of $\mathbf{A}_1$ and $\mathbf{A}_2$. Then, [KL14, Proposition 4.6] gives a semiorthogonal decomposition of the derived category:

$$\mathsf{D}(\mathbf{A}) = \langle \mathsf{D}(\mathbf{A}_1), \mathsf{D}(\mathbf{A}_2)\rangle,$$

with inclusion functors $I_k = \mathbb{L}\operatorname{Ind}_{i_k}$, where $i_k \colon \mathbf{A}_k \to \mathbf{A}$ is the canonical inclusion, for $k = 1, 2$. For any $\mathbf{A}$-module M, there is a canonical distinguished triangle

$$I_2 I_2^!(M) \to M \to I_1 I_1^*(M).$$

I_1 and I_2 preserve quasi-representable modules, so since $\mathbf{A}$ is pretriangulated we find out that M is quasi-representable if $I_2^!(M)$ and $I_1^*(M)$ are quasi-representable. Conversely, let $M \overset{\mathrm{qis}}{\approx} h_X$. Then, since $H^0(\mathbf{A}) = \langle H^0(\mathbf{A}_1), H^0(\mathbf{A}_2)\rangle$, there is a canonical distinguished triangle in $\mathsf{D}(\mathbf{A})$:

$$h_{i_2(X_2)} \to h_X \to h_{i_1(X_1)}.$$

This triangle has to be functorially associated to M, so we have that $I_1^*(M) \overset{\mathrm{qis}}{\approx} h_{X_1}$ and $I_2^!(M) \overset{\mathrm{qis}}{\approx} h_{X_2}$, and they are quasi-representable. Now, to obtain our desired result, we must show that any direct summand M in $\mathsf{D}(\mathbf{A})$ of a quasi-representable $\mathbf{A}$-module is quasi-representable. But $I_1^*(M)$ and $I_2^!(M)$ are themselves direct summands of quasi-representable dg-modules, hence they are quasi-representable, and we conclude that M itself is quasi-representable.

We now come to the dg-lift uniqueness problem. We are able to obtain positive results in some cases where the domain dg-category has a full and strong exceptional sequence. It is known that $\mathrm{per}(\mathbf{k})$ is a triangulated category generated by the exceptional object $\mathbf{k}$ (viewed as complex concentrated in degree 0), which is the only indecomposable object (up to shift): every object in $\mathrm{per}(\mathbf{k})$ is obtained from $\mathbf{k}$ by means of shifts and finite direct sums. So, we are able to prove the following:

Lemma 4.3.3. *Let* $\mathbf{B}$ *be a triangulated dg-category. The functor*

$$\Phi^{\mathrm{per}_{\mathrm{dg}}(\mathbf{k})\to\mathbf{B}} \colon H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathrm{per}_{\mathrm{dg}}(\mathbf{k}),\mathbf{B})) \to \mathrm{Fun}_{\mathrm{ex}}(\mathrm{per}(\mathbf{k}), H^0(\mathbf{B}))$$

is an equivalence.

Proof. As in the proof of Lemma 3.6.5, we have the following commutative diagram:

$$\begin{array}{ccc} H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathrm{per}_{\mathrm{dg}}(\mathbf{k}),\mathbf{B})) & \xrightarrow{\Phi^{\mathrm{per}_{\mathrm{dg}}(\mathbf{k})\to\mathbf{B}}} & \mathrm{Fun}_{\mathrm{ex}}(H^0(\mathrm{per}_{\mathrm{dg}}(\mathbf{k})), H^0(\mathbf{B})) \\ \downarrow{\sim} & & \downarrow \\ H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{k},\mathbf{B})) & \xrightarrow{\Phi^{\mathbf{k}\to\mathbf{B}}} & \mathrm{Fun}(\mathbf{k}, H^0(\mathbf{B})), \end{array}$$

The functor $\Phi^{\mathbf{k}\to\mathbf{B}}$ is an equivalence by Lemma 3.6.4. Also, the right vertical arrow is an equivalence: since any object of $H^0(\mathrm{per}_{\mathrm{dg}}(\mathbf{k})) \cong \mathrm{per}(\mathbf{k})$ is of the form $\bigoplus_{i=1}^N \mathbf{k}[n_i]^{\oplus m_i}$, we see that any exact functor $\mathrm{per}(\mathbf{k}) \to \mathbf{B}$ is determined by its value on $\mathbf{k}$, and the same is true for natural transformations. We conclude that $\Phi^{\mathrm{per}_{\mathrm{dg}}(\mathbf{k})\to\mathbf{B}}$ is an equivalence, as claimed. □

The above result tells us that the dg-lift uniqueness problem is completely solved, in the trivial case of exceptional sequences of length 1: if $\mathbf{A}$ is a triangulated dg-category with a single exceptional object E_0, then by Lemma 4.3.1 we have that $\mathbf{A} \overset{\mathrm{qe}}{\approx} \mathrm{per}_{\mathrm{dg}}(\mathbf{k})$. Now, we would like to address the problem for dg-categories with full and strong exceptional sequences of length greater than 1.

The free case

Let $\mathbf{A}$ be a triangulated and locally perfect dg-category, admitting a two-term full and strong exceptional sequence (E_0, E_1). By Lemma 4.3.1, we have that $\mathbf{A} \overset{\mathrm{qe}}{\approx} \mathrm{per}_{\mathrm{dg}}(\mathbb{E})$, where $\mathbb{E}$ is a $\mathbf{k}$-linear category with two objects E_0, E_1 and a finite-dimensional hom-space (we are under the local properness assumption). We immediately notice that $\mathbb{E}$ is *free*, namely, it is isomorphic to the free $\mathbf{k}$-category $\mathbf{k}[Q]$ over the quiver Q with objects E_0, E_1 and hom-space $\mathbb{E}(E_0, E_1)$. The dg-lift uniqueness problem, in the free case, has a positive solution. The technique is based on homotopies of dg-functors, as explained in Section 1.3. First, let us give a precise definition of free $\mathbf{k}$-category generated by a quiver:

Definition 4.3.4. Let Q be a (small) quiver. The category $\mathbf{k}[Q]$ freely generated by Q over $\mathbf{k}$ is the $\mathbf{k}$-linear category defined as follows:

- $\operatorname{Ob}(\mathbf{k}[Q]) = \operatorname{Ob} Q$.
- For $a, b \in Q$, $\mathbf{k}[Q](a, b)$ is the free $\mathbf{k}$-module generated by all finite strings of composable arrows with initial source a and final target b. Identities are the empty strings. Compositions are given by string concatenation, extended by bilinearity.

Proposition 4.3.5. *For any quiver Q, the category $\mathbf{k}[Q]$, viewed as a dg-category, is a cofibrant object in* dgCat.

In order to prove this, we need a preparatory lemma:

Lemma 4.3.6. *Let $p\colon \widetilde{V} \to V$ be a surjective quasi-isomorphism in $\mathsf{C}(\mathbf{k})$.*

1. *$Z^*(p)\colon Z^*(\widetilde{V}) \to Z^*(V)$ is surjective. In other words, closed elements of V can be lifted to closed elements of $\widetilde{V}$.*
2. *Let $x, y \in V$ be such that $x = dy$. Let $\tilde{x} \in p^{-1}(x)$. Then, there exists $\tilde{y} \in p^{-1}(y)$ such that $\tilde{x} = d\tilde{y}$.*

Proof. *(1)* Let $x \in Z^*(V)$ be a closed element. Consider $[x] \in H^*(V)$. Then, there exists $\tilde{x} \in Z^*(\widetilde{V})$ such that $p(\tilde{x}) = x + dz$, for some $z \in V$, because $H^*(p)$ is surjective. p is surjective, so there exists $\tilde{z} \in \widetilde{V}$ such that $p(\tilde{z}) = z$. Hence, we find that $p(\tilde{x}) = x + dz = x + p(d\tilde{z})$, so in the end $\tilde{x} - d\tilde{z} \in Z^*(\widetilde{V})$ is an element of $p^{-1}(x)$.

(2) By hypothesis, $H^*(p)\colon H^*(\widetilde{V}) \to H^*(V)$ is an isomorphism. We have:

$$H^*(p)[\tilde{x}] = [p(\tilde{x})] = [x] = [dy] = [0],$$

hence $[\tilde{x}] = [0]$, that is, there exists $\tilde{y}_1 \in \widetilde{V}$ such that $\tilde{x} = d\tilde{y}_1$. Now,

$$p(\tilde{x}) = x = dy = dp(\tilde{y}_1),$$

so $y - p(\tilde{y}_1)$ is closed. By part *(1)*, there exists $\tilde{y}_2 \in Z(\widetilde{V})$ such that $p(\tilde{y}_2) = y - p(\tilde{y}_1)$. Finally, set

$$\tilde{y} = \tilde{y}_1 + \tilde{y}_2.$$

By construction, $p(\tilde{y}) = y$, and $d\tilde{y} = d\tilde{y}_1 + d\tilde{y}_2 = d\tilde{y}_1 = \tilde{x}$. □

Proof (Proposition 4.3.5). Let $\mathbf{A}, \mathbf{B}$ be small dg-categories, let $F\colon \mathbf{k}[Q] \to \mathbf{B}$ be a dg-functor, and let $G\colon \mathbf{A} \to \mathbf{B}$ be a trivial fibration in dgCat. We have to find a dg-functor $\widetilde{F}\colon \mathbf{k}[Q] \to \mathbf{A}$ such that $G\widetilde{F} = F$.

By Remark 1.3.4, G is surjective on objects. Hence, we can find a lift $\widetilde{F}$ of F on objects: $G\widetilde{F}(a) = F(a)$ for all $a \in Q$.

Next, we lift F on morphisms, in order to obtain a dg-functor $\widetilde{F}$. Since $\mathbf{k}[Q]$ is freely generated by Q, it suffices to map any arrow $f\colon a \to b$ of Q to a *closed degree* 0 morphism $\widetilde{F}(f)$ such that $G\widetilde{F}(f) = F(f)$. This can be done by Lemma 4.3.6, part 1. □

Now, we have the tools to prove the dg-lift uniqueness result in the free case. The actual argument of proof is very simple:

Lemma 4.3.7. *Let Q be a quiver, let $\mathbf{B}$ be a dg-category, and let $F, G\colon k[Q] \to \mathbf{B}$ be dg-functors. Let $\overline{\varphi}\colon H^0(F) \to H^0(G)$ be a natural transformation. Then, there exists a directed homotopy $\varphi\colon F \to G$ which induces $\overline{\varphi}$ (recall Remark 1.3.14). Moreover, φ is a homotopy if $\overline{\varphi}$ is a natural isomorphism.*

Proof. By hypothesis, for any $f\colon a \to b$ in Q, there is a commutative diagram in $H^0(\mathbf{B})$:

$$\begin{array}{ccc} F(a) & \xrightarrow{\overline{\varphi}_a} & G(a) \\ {\scriptstyle [F(f)]}\downarrow & & \downarrow{\scriptstyle [G(f)]} \\ F(b) & \xrightarrow{\overline{\varphi}_b} & G(b). \end{array}$$

We use these data to define a directed homotopy $\varphi\colon \mathbf{k}[Q] \to \underline{\mathrm{Mor}}\,\mathbf{B}$ from F to G. To define it on objects, we choose for any $a \in Q$ a representative $\varphi_a\colon F(a) \to G(a)$ of $\overline{\varphi}_a$, and set

$$\varphi(a) = (F(a), G(a), \varphi_a).$$

Next, since $\mathbf{k}[Q]$ is freely generated by Q, it is sufficient to map any arrow of Q to a closed degree 0 morphism in $\underline{\mathrm{Mor}}\,\mathbf{B}$. To do so, we choose for any $f\colon a \to b$ in Q a morphism $h(f) \in \mathbf{B}(F(a), G(b))^{-1}$ such that

$$dh(f) = G(f)\varphi_a - \varphi_b F(f),$$

and we set

$$\varphi(f) = (F(f), G(f), h(f)).$$

This definition extends to $\mathbf{k}[Q]$ and gives the desired directed homotopy: by construction, $H^0(\varphi) = \overline{\varphi}$. If $\overline{\varphi}$ is a natural isomorphism, then we have already observed in general that φ takes values in $P(\mathbf{B})$, and it is a right homotopy (see Remark 1.3.14). □

From this, we finally obtain the desired dg-lift uniqueness result:

Theorem 4.3.8. *Let $\mathbf{A}$ and $\mathbf{B}$ be triangulated dg-categories. Assume that $\mathbf{A}$ is generated by a free $\mathbf{k}$-linear category of the form $\mathbf{k}[Q]$, for some quiver Q. Then, $\Phi^{\mathbf{A}\to\mathbf{B}}$ is essentially injective.*

Proof. By Lemma 4.3.1, we have that $\mathbf{A}$ is quasi-equivalent to $\mathrm{per}_{\mathrm{dg}}(\mathbf{k}[Q])$, so by Lemma 3.6.5 it is sufficient to prove that $\Phi^{\mathbf{k}[Q]\to\mathbf{B}}$ is essentially injective. Let $\widehat{F}, \widehat{G} \in \mathsf{Hqe}(\mathbf{k}[Q], \mathbf{B})$. Since $\mathbf{k}[Q]$ is cofibrant, they are represented by actual dg-functors F and G. Now, Lemma 4.3.7 tells us that $H^0(F) \cong H^0(G)$ implies that F is homotopic to G, hence $\widehat{F} = \widehat{G}$ in $\mathsf{Hqe}(\mathbf{A}, \mathbf{B})$, by Corollary 1.3.13. □

This result has a geometric application. Let X be a quasi-projective scheme, and let $\mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(X))$ be an enhancement of the derived category $\mathfrak{D}(\mathrm{QCoh}(X))$ of quasi-coherent sheaves on X. This category is idempotent complete; for simplicity identify it to $H^0(\mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(X)))$. The full dg-subcategory $\mathrm{Perf}_{\mathrm{dg}}(X)$ of $\mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(X))$ whose objects are the compact objects in $\mathfrak{D}(\mathrm{QCoh}(X))$ is an enhancement of the category of perfect complexes $\mathrm{Perf}(X)$. These enhancements are uniquely determined up to quasi-equivalence, by [LO10, Corollary 7.8, Theorem 7.9]. If $X = \mathbb{P}^n$, then $\mathrm{Perf}(X)$ has a strong and full exceptional sequence:

Theorem 4.3.9 ([Beĭ78])**.** *The category* $\mathrm{Perf}(\mathbb{P}^n)$ *admits a strong and full exceptional sequence* $(\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n))$*, with the property that for any* $0 \leq d \leq d' \leq n$*, the hom-space* $\mathrm{Perf}(X)(\mathcal{O}(d), \mathcal{O}(d'))$ *is isomorphic to the vector space* $S_{d'-d}$ *of homogeneous polynomials of degree* $d' - d$ *in* $n+1$ *indeterminates.*

In particular, $\mathrm{Perf}(\mathbb{P}^1)$ has a two-term strong and full exceptional sequence $(\mathcal{O}, \mathcal{O}(1))$, and the hom-space $\mathrm{Perf}(\mathbb{P}^1)(\mathcal{O}, \mathcal{O}(1))$ has dimension 2. So, Theorem 4.3.8 is applicable, and yields:

Corollary 4.3.10. *Let* $\mathbf{B}$ *be a triangulated dg-category. Let* $F, G \colon \mathrm{Perf}_{\mathrm{dg}}(\mathbb{P}^1) \to \mathbf{B}$ *be quasi-functors. Then, if* $H^0(F) \cong H^0(G)$*, we have that* $F \cong G$ *as quasi-functors.*

Now, using the results of Section 0.2, we obtain the following uniqueness result of Fourier-Mukai kernels:

Corollary 4.3.11. *Let* X *and* Y *be schemes satisfying the hypotheses of both Theorems 0.2.2 and 0.2.1, with* $X = \mathbb{P}^1$*. Let* $\mathcal{E}, \mathcal{E}' \in \mathfrak{D}(\mathrm{QCoh}(X \times Y))$ *be such that*

$$\Phi_{\mathcal{E}}^{X \to Y} \cong \Phi_{\mathcal{E}'}^{X \to Y} \colon \mathrm{Perf}(X) \to \mathfrak{D}(\mathrm{QCoh}(Y)),$$

Then $\mathcal{E} \cong \mathcal{E}'$*.*

The above result was proved by Canonaco and Stellari with geometric techniques, see [CS12a, Lemma 4.2].

The glueing technique

Now, we attempt to solve the dg-lift uniqueness problem for dg-categories which have exceptional sequences with arbitrary length. The next result will allow us to apply the technique explained in Corollary 4.2.10.

Proposition 4.3.12. *Let* $\mathbf{A}$ *be a locally proper and triangulated dg-category, and assume that it admits a full exceptional sequence* ($n \geq 0$)*:*

$$H^0(\mathbf{A}) = \langle E_0, \dots E_n \rangle.$$

Then, $\mathbf{A}$ *is quasi-equivalent to the glueing* $\mathbf{A}_{0,\dots,n-1} \times_N \mathbf{A}_n$*, where* $\mathbf{A}_{0,\dots,n-1}$ *is the full dg-subcategory of* $\mathbf{A}$ *whose objects are the same as the triangulated subcategory*

$\langle E_0, \dots E_{n-1} \rangle$, and analogously $\mathbf{A}_n$ is the full dg-subcategory whose objects are the same as that of $\langle E_n \rangle$; N is obtained from the diagonal bimodule by restriction and shift:

$$N(A', A'') = \mathbf{A}(i_1(A'), i_2(A'')[1]),$$

where i_1 and i_2 are the obvious inclusion dg-functors. Moreover, $\mathbf{A}$ is smooth and N is a quasi-functor.

Proof. If $n = 0$, by Lemma 4.3.1 we have that $\mathbf{A} \overset{\text{qe}}{\approx} \operatorname{per}_{\text{dg}}(\mathbf{k})$, which is smooth since $\mathbf{k}$ is smooth viewed as a dg-category (see Remark 3.5.7). Assume $n > 0$. The glueing decomposition of $\mathbf{A}$ follows directly from Proposition 4.2.5. Arguing by induction, assume that $\mathbf{A}' = \mathbf{A}_{0,\dots,n-1}$ is smooth. Both $\mathbf{A}'$ and $\mathbf{A}_n$ are triangulated, being enhancements of thick subcategories of $H^0(\mathbf{A})$ (recall Remark 4.1.2); $\mathbf{A}_n$ is smooth, being quasi-equivalent to $\operatorname{per}_{\text{dg}}(\mathbf{k})$. Now, applying Lemma 3.5.8, $N \colon \mathbf{A}_n \rightsquigarrow \mathbf{A}'$ is a perfect bimodule if and only if N_{A_n} is a perfect right $\mathbf{A}'$-module for all $A_n \in \mathbf{A}_n$. Now, view N_{A_n} as a bimodule $\mathbf{A}'^{\text{op}} \rightsquigarrow \mathbf{k}$: again by Lemma 3.5.8, N_{A_n} is perfect if and only if $N_{A_n}^{A'}$ is a perfect $\mathbf{k}$-module for all $A' \in \mathbf{A}'$, and this is true by the local properness assumption. Hence, N is perfect, and by Proposition 4.9 of [KL14] we deduce that $\mathbf{A}$ is smooth. Moreover, since $\mathbf{A}'$ is triangulated and N_{A_n} is perfect for all $A_n \in \mathbf{A}_n$, we deduce that N_{A_n} is quasi-representable for all A_n, that is, N is a quasi-functor. □

In the above framework, we know that $H^0(\mathbf{A})$ has a semiorthogonal decomposition $\langle \{E_0, \dots E_{n-1}\}, E_n \rangle$. Since $N \colon \mathbf{A}_n \to \mathbf{A}'$ is a quasi-functor, it induces a glueing functor, which we call again N abusing notation, defined by the natural isomorphism

$$H^0(\mathbf{A})(i_1(A'), i_2(A'')[1]) \cong H^0(\mathbf{A}')(A', N(A'')). \tag{4.3.1}$$

Now, since $\mathbf{A}_n = \langle E_n \rangle \overset{\text{qe}}{\approx} \operatorname{per}_{\text{dg}}(\mathbf{k})$, we have by Lemma 4.3.3 that the quasi-functor N is completely determined by the glueing functor N in homotopy, which in turn is completely determined by the value $N(E_n)$. Clearly, this observation holds for any quasi-functor defined on $\mathbf{A}_n$. Moreover, the counit

$$\varepsilon \colon H^0(i_1) \circ N[-1] \to H^0(i_2)$$

of the isomorphism (4.3.1) is the natural transformation induced in cohomology by the adjunction of quasi-functors $i_1 \dashv i_1^!$. Again, by Lemma 4.3.3, we may identify it to the morphism

$$\alpha = \varepsilon_{E_n} \colon i_1 N(E_n[-1]) \to i_2(E_n). \tag{4.3.2}$$

Now, let $F \colon \mathbf{A} \to \mathbf{B}$ be a quasi-functor, with $\mathbf{B}$ triangulated. By the above discussion, it should now be clear that the morphism $(F \diamond^{\mathbb{L}} i_1) \diamond^{\mathbb{L}} N[-1] \to F \diamond^{\mathbb{L}} i_2$ in the triangle (4.2.5) can be identified with

$$H^0(F)(\alpha) \colon F i_1 N(E_n[-1]) \to F i_2(E_n). \tag{4.3.3}$$

So, Corollary 4.2.10 boils down to the following:

Corollary 4.3.13. *Let* $\mathbf{A}$ *be as in Proposition 4.3.12. Let* $\mathbf{B}$ *be a triangulated dg-category, and let* $F, G \colon \mathbf{A} \to \mathbf{B}$ *be quasi-functors. Let* $\varphi_1 \colon F \diamond^{\mathbb{L}} i_1 \to G \diamond^{\mathbb{L}} i_1$ *be a morphism of quasi-functors and* $\varphi_2' \colon F(i_2(E_n)) \to G(i_2(E_n))$ *be a morphism in* $H^0(\mathbf{B})$*. Assume that the diagram*

$$\begin{array}{ccc} Fi_1N(E_n[-1]) & \xrightarrow{H^0(F)(\alpha)} & Fi_2(E_n) \\ \Big\downarrow{\scriptstyle H^0(\varphi_1)_{N(E_n[-1])}} & & \Big\downarrow{\scriptstyle \varphi_2'} \\ Gi_1N(E_n[-1]) & \xrightarrow{H^0(G)(\alpha)} & Gi_2(E_n) \end{array} \tag{4.3.4}$$

is commutative in $H^0(\mathbf{B})$*. Complete this to a morphism of distinguished triangles in* $H^0(\mathbf{B})$*:*

$$\begin{array}{ccccccc} Fi_1N(E_n[-1]) & \xrightarrow{H^0(F)(\alpha)} & Fi_2(E_n) & \longrightarrow & F(\mathrm{C}(\alpha)) & \xrightarrow{\psi_F} & Fi_1N(E_n) \\ \Big\downarrow{\scriptstyle H^0(\varphi_1)_{N(E_n[-1])}} & & \Big\downarrow{\scriptstyle \varphi_2'} & & \Big\downarrow{\scriptstyle \varphi_2} & & \Big\downarrow{\scriptstyle H^0(\varphi_1)_{N(E_n)}} \\ Gi_1N(E_n[-1]) & \xrightarrow{H^0(G)(\alpha)} & Gi_2(E_n) & \longrightarrow & G(\mathrm{C}(\alpha)) & \xrightarrow{\psi_G} & Gi_1N(E_n); \end{array}$$

then, there exists a morphism $\varphi \colon F \to G$ *of quasi-functors such that* $H^0(\varphi)_{\mathrm{C}(\alpha)} = \varphi_2$*, and* $\varphi \diamond^{\mathbb{L}} i_1 = \varphi_1$*. Moreover, if* φ_1 *and* φ_2 *are isomorphisms, so is* φ*.*

Remark 4.3.14. It is essential that the arrow $Fi_1N(E_n) \to Gi_1N(E_n)$ in the above commutative diagram comes from a morphism of quasi-functors φ_1; otherwise, the argument doesn't work, as we will see in a counterexample in the next section.

In the glueings of the above form $\mathbf{A}_{0,\dots,n-1} \times_N \mathbf{A}_n$, the quasi-functor N is determined by the object $N(E_n) = C$, therefore we will simplify notation and write $\mathbf{A}_{0,\dots,n-1} \times_C \mathbf{A}_n$. Now, the idea is to apply Corollary 4.3.13 iteratively, in order to deduce dg-lift uniqueness results for categories with exceptional sequences of any length. Unfortunately, this technique is not very successful, in fact it doesn't give us a result for any exceptional sequence; even worse, the cases which are relevant in geometric applications remain unsolved. However, it is sufficient to show that dg-lift uniqueness holds in some cases even if the category of generators is not free. By working out the argument, we will see what is its weak point.

We start from a simple free case which we already treated, namely, we assume that $\mathbf{A} = \mathrm{per}_{\mathrm{dg}}(\Delta^1)$, where Δ^1 is the standard 1-simplex category, freely generated by the diagram $E_0 \xrightarrow{e_0} E_1$. From Remark 4.3.2, we know that $\mathbf{A}$ is quasi-equivalent to $\mathrm{pretr}(\Delta^1)$; so, $\mathbf{A}$ is a triangulated dg-category with a full and strong exceptional sequence $\{E_0, E_1\}$. It can be proved that $H^0(\mathbf{A})$ has three indecomposable objects, that is, $E_0, E_1, \mathrm{C}(e_0)$. Any object of $H^0(\mathbf{A})$ can be obtained by the indecomposable objects by means of shifts and direct sums; moreover, a natural transformation of exact functors $\varphi \colon F \to G$ from $H^0(\mathbf{A})$ to another triangulated category $\mathbf{T}$ is completely determined by the maps φ_{E_0}, φ_{E_1} and $\varphi_{\mathrm{C}(e_0)}$ together with the compatibilities expressed by the following commutative diagram:

$$\begin{array}{ccccccc} F(E_0) & \xrightarrow{F(e_0)} & F(E_1) & \xrightarrow{F(j_0)} & F(\mathrm{C}(e_0)) & \xrightarrow{F(p_0)} & F(E_0)[1] \\ \Big\downarrow{\scriptstyle \varphi_{E_0}} & & \Big\downarrow{\scriptstyle \varphi_{E_1}} & & \Big\downarrow{\scriptstyle \varphi_{\mathrm{C}(e_0)}} & & \Big\downarrow{\scriptstyle \varphi_{E_0}[1]} \\ G(E_0) & \xrightarrow{G(e_0)} & G(E_1) & \xrightarrow{G(j_0)} & G(\mathrm{C}(e_0)) & \xrightarrow{G(p_0)} & G(E_0)[1], \end{array} \tag{4.3.5}$$

where j_0 and p_0 are the maps obtained in the triangle which extends e_0:

$$E_0 \xrightarrow{e_0} E_1 \xrightarrow{j_0} \mathrm{C}(e_0) \xrightarrow{p_0} E_0[1].$$

It is worth noticing that $H^0(\mathbf{A})$ has other full and strong exceptional sequences, other than $\{E_0, E_1\}$. Indeed, we have:

$$H^0(\mathbf{A}) = \langle E_0, E_1 \rangle = \langle E_1, \mathrm{C}(e_0) \rangle = \langle \mathrm{C}(e_0), E_0[1] \rangle.$$

So, by Proposition 4.3.12, we may identify $\mathbf{A}$, up to quasi-equivalence, to either of the following glueings:

$$\begin{aligned} \mathbf{A} &\overset{\text{qe}}{\approx} \langle E_0 \rangle \times_{C_1} \langle E_1 \rangle \\ &\overset{\text{qe}}{\approx} \langle E_1 \rangle \times_{C_2} \langle \mathrm{C}(e_0) \rangle \\ &\overset{\text{qe}}{\approx} \langle \mathrm{C}(e_0) \rangle \times_{C_3} \langle E_0[1] \rangle, \end{aligned}$$

where, abusing notation, we have denoted $\langle X \rangle$ the full dg-subcategory of $\mathbf{A}$ whose objects are the same as the triangulated subcategory of $H^0(\mathbf{A})$ generated by X. A simple computation gives:

$$\begin{aligned} C_1 &= E_0[1], \\ C_2 &= E_1[1], \\ C_3 &= \mathrm{C}(e_0)[1]. \end{aligned}$$

For example, C_1 is characterised to be the object of $\langle E_0 \rangle$ such that there is a natural isomorphism

$$H^0(\mathbf{A})(E_0, E_1[1]) \cong \langle E_0 \rangle (E_0, C_1),$$

and we immediately see that it is necessarily $E_0[1]$. This isomorphism maps the identity $1_{E_0[1]}$ to $e_0[1] \colon E_0[1] \to E_1[1]$, so we deduce that the morphism (4.3.2) is indeed given by $e_0 \colon E_0 \to E_1$. Analogously, in the other situations we find the maps $j_0 \colon E_1 \to \mathrm{C}(e_0)$ and $p_0 \colon \mathrm{C}(e_0) \to E_0[1]$.

Now, let $\mathbf{B}$ be a triangulated dg-category, and let $F, G \colon \mathbf{A} \to \mathbf{B}$ be quasi-functors. Assume we are given an isomorphism $\overline{\varphi} \colon H^0(F) \to H^0(G)$. By Lemma 4.3.3, we may identify the objects $F(E_0), G(E_0)$ and so on to quasi-functors obtained from F and G by restriction. Also, we may view $\overline{\varphi}_{E_0}$, $\overline{\varphi}_{E_1}$ and $\overline{\varphi}_{\mathrm{C}(e_0)}$ as morphisms of quasi-functors.

The hypothesis gives us the following commutative diagram:

$$\begin{array}{ccccccc} F(E_0) & \xrightarrow{H^0(F)(e_0)} & F(E_1) & \xrightarrow{H^0(F)(j_0)} & F(\mathrm{C}(e_0)) & \xrightarrow{H^0(F)(p_0)} & F(E_0)[1] \\ \downarrow{\scriptstyle \overline{\varphi}_{E_0}} & & \downarrow{\scriptstyle \overline{\varphi}_{E_1}} & & \downarrow{\scriptstyle \overline{\varphi}_{\mathrm{C}(e_0)}} & & \downarrow{\scriptstyle \overline{\varphi}_{E_0}[1]} \\ G(E_0) & \xrightarrow{H^0(G)(e_0)} & G(E_1) & \xrightarrow{H^0(G)(j_0)} & G(\mathrm{C}(e_0)) & \xrightarrow{H^0(G)(p_0)} & G(E_0)[1]. \end{array}$$

Now, it is clear that we can apply Corollary 4.3.13 to $\mathbf{A}$, viewed as generated by one of the three exceptional sequences $\{E_0, E_1\}$, $\{E_1, \mathrm{C}(e_0)\}$ or $\{\mathrm{C}(e_0), E_0[1]\}$. In the end, what we obtain is the following:

Lemma 4.3.15. *Let $\mathbf{A}$ be a triangulated and locally perfect dg-category, generated by a two-term strong and full exceptional sequence with 1-dimensional hom-space. Let $\mathbf{B}$ be another triangulated dg-category, let $F, G \colon \mathbf{A} \to \mathbf{B}$ be quasi-functors, and let $\overline{\varphi} \colon H^0(F) \to H^0(G)$ be a natural transformation. Then, for any two-element subset S of $\{E_0, E_1, \mathrm{C}(e_0)\} \subseteq \mathbf{A}$, there exists a morphism of quasi-functors $\varphi \colon F \to G$ such that $H^0(\varphi)$ is equal to $\overline{\varphi}$ on S. Moreover, if $\overline{\varphi}$ is an isomorphism, then so is φ.*

So, we don't lift a given natural transformation $\overline{\varphi}$ as above to a morphism of quasi-functors (it is actually *impossible* to do so in general, as we will see in the next section), but we may find a "partial lift". This enables us to go on and prove dg-lift uniqueness for dg-categories with particular three-term exceptional sequences, namely, dg-categories obtained glueing along "selected objects". We just give some sketches. Let $\mathbf{A}'$ be a triangulated dg-category generated by a two-term strong and full exceptional sequence with 1-dimensional hom-space, as in the previous discussion. Consider

$$\mathbf{A} = \mathbf{A}' \times_C \langle E_2 \rangle, \tag{4.3.6}$$

where $\langle E_2 \rangle$ is $\mathrm{per}_{\mathrm{dg}}(\mathbf{k})$, where $\mathbf{k}$ is notationally identified with E_2. Denote by i_k the natural inclusion dg-functors ($k = 1, 2$). Assume that C is an object among $\{E_0, E_1, \mathrm{C}(e_0)\}$ or is obtained as direct sums of (shifts of) at most *two* of such objects. For example, take $C = \mathrm{C}(e_0)$, or $C = E_0^{\oplus 7} \oplus E_1[4]$, and so on. Let $\mathbf{B}$ a triangulated dg-category, and let $F, G \colon \mathbf{A} \to \mathbf{B}$ be quasi-functors. Assume we are given $\overline{\varphi} \colon H^0(F) \to H^0(G)$. Then, consider $\overline{\varphi} \circ i_1 \colon H^0(Fi_1) \to H^0(Gi_1)$; by the above Lemma 4.3.15, we may find a morphism of quasi-functors $\varphi_1 \colon F \diamond^{\mathbb{L}} i_1 \to G \diamond^{\mathbb{L}} i_1$ such that $H^0(\varphi_1)_C = \overline{\varphi}_{i_1(C)}$: this is because φ_1 can be found as a lift of $\overline{\varphi} \circ i_1$ on any couple of objects among $\{E_0, E_1, \mathrm{C}(e_0)\}$, and hence it must be a lift also on objects as C obtained with direct sums and shifts of at most two such objects. Now, the hypothesis gives us a commutative diagram:

$$\begin{array}{ccccc} Fi_1(C[-1]) & \xrightarrow{H^0(F)(\alpha)} & Fi_2(E_2) & \longrightarrow & F(\mathrm{C}(\alpha)) \\ \downarrow{\scriptstyle \overline{\varphi}_{i_1(C)[-1])} = H^0(\varphi_1)_{C[-1]}} & & \downarrow{\scriptstyle \overline{\varphi}_{i_2(E_2)}} & & \downarrow{\scriptstyle \overline{\varphi}_{\mathrm{C}(\alpha)}} \\ Gi_1N(E_n[-1]) & \xrightarrow{H^0(G)(\alpha)} & Gi_2(E_2) & \longrightarrow & G(\mathrm{C}(\alpha)). \end{array}$$

So, applying Corollary 4.3.13, we find a morphism $\varphi \colon F \to G$ of quasi-functors such that $\varphi \diamond^{\mathbb{L}} i_1 = \varphi_1$ and $H^0(\varphi)_{\mathrm{C}(\alpha)} = \overline{\varphi}_{\mathrm{C}(\alpha)}$. It is an isomorphism if $\overline{\varphi}$ is such.

Now, it should be clear how we can go on. We shall glue $\mathbf{A} = \mathbf{A}' \times_C \langle E_2 \rangle$ to $\langle E_3 \rangle$ along $\mathrm{C}(\alpha)$, and get another "partial lifting" result. So, we obtain dg-lift uniqueness for dg-categories with exceptional sequences of arbitrary length, but of very particular nature. The result is unsatisfactory, and in some sense it shows us that the dg-lift uniqueness problem is very difficult even for dg-categories of very simple kind.

4.4 Counterexamples (attempts)

As we proved, the dg-lift uniqueness holds when the domain dg-category is generated by a free $\mathbf{k}$-linear category. The "glueing technique" explained in the previous discussion gives some examples where the result holds even if the domain dg-category is generated by a non-free subcategory. However, it should be pointed out that the result in the free case is somewhat stronger: it tells us that the functor $\Phi^{\mathbf{k}[Q] \to \mathbf{B}}$ is essentially injective, from which we deduce that $\Phi^{\mathrm{per}_{\mathrm{dg}}(\mathbf{k}[Q]) \to \mathbf{B}}$ is such, if $\mathbf{B}$ is a triangulated dg-category. Actually, there exist dg-categories $\mathbf{D}$ such that $\Phi^{\mathrm{per}_{\mathrm{dg}}(\mathbf{D}) \to \mathbf{B}}$ is essentially injective but $\Phi^{\mathbf{D} \to \mathbf{B}}$ is not, as we are going to show with the following counterexample.

A "semi-counterexample" to uniqueness

Consider the simplest non-free category which serves as exceptional sequence, namely, the category $\mathbf{D}$ with three objects E_0, E_1, E_2, freely generated over $\mathbf{k}$ by the following diagram:

$$\begin{array}{ccc} E_0 & \xrightarrow{e_0} & E_1 \\ & \searrow^{0} & \downarrow{\scriptstyle e_1} \\ & & E_2, \end{array}$$

with the unique relation $e_1 e_0 = 0$. Now, the category $\mathbf{A} = \mathrm{pretr}(\mathbf{D})$ is pretriangulated and generated by the exceptional sequence $\{E_0, E_1, E_2\}$. As we know, it can be viewed as a glueing of the form $\langle E_0, E_1 \rangle \times_C \langle E_2 \rangle$, where here $\langle E_0, E_1 \rangle \overset{\mathrm{qe}}{\approx} \mathrm{pretr}(\Delta^1) \overset{\mathrm{qe}}{\approx} \mathrm{per}_{\mathrm{dg}}(\Delta^1)$, and $\langle E_2 \rangle \overset{\mathrm{qe}}{\approx} \mathrm{per}_{\mathrm{dg}}(\mathbf{k})$. To understand what is $C \in \langle E_0, E_1 \rangle$, we recall that it is defined by the natural isomorphism

$$H^0(\mathbf{A})(i_1(-), i_2(E_2)[1]) \cong \langle E_0, E_1 \rangle(-, C).$$

Then, a simple inspection shows that $C = \mathrm{C}(e_0)[1]$.

The previous discussion on dg-categories of the form (4.3.6) shows that the dg-lift uniqueness holds true in this case, that is, the functor $\Phi^{\mathbf{A} \to \mathbf{B}}$ is essentially injective for all (pre)triangulated dg-categories $\mathbf{B}$. What about $\Phi^{\mathbf{D} \to \mathbf{B}}$? The answer is *no*. To see this, we employ homotopies, as we did for the result in the free case. The category $\mathbf{D}$ is not cofibrant, as we will see, but there is a rather standard technique that produces a cofibrant replacement $\widetilde{\mathbf{D}}$. The idea is that any nontrivial relation, such has $e_1 e_0 = 0$, should be substituted with a "homotopy coherent" variant, namely, something like $e_1 e_0 = d e_{01}$ for some degree -1 morphism e_{01}. Let us make this idea more precise.

Example 4.4.1. We consider the dg-category $\widetilde{\mathbf{D}}$ with three objects E_0, E_1, E_2 (the same object set as $\mathbf{D}$), and morphisms freely generated by $\tilde{e}_0 \in \mathbf{D}(E_0, E_1)^0$, $\tilde{e}_1 \in \mathbf{D}(E_1, E_2)^0$ and $\tilde{e}_{01} \in \mathbf{D}(E_0, E_2)^{-1}$, with the only nontrivial differential defined on $\mathbf{D}(E_0, E_2)$ by $d\tilde{e}_{01} = \tilde{e}_1\tilde{e}_0$:

$$\begin{array}{ccc} E_0 & \xrightarrow{\tilde{e}_0} & E_1 \\ & \searrow^{\tilde{e}_{01}} & \downarrow{\scriptstyle \tilde{e}_1} \\ & & E_2. \end{array} \tag{4.4.1}$$

Now, we prove that the dg-category $\widetilde{\mathbf{D}}$ is cofibrant. Let $G\colon \mathbf{A} \to \mathbf{B}$ be a trivial fibration in dgCat, and let $F\colon \widetilde{\mathbf{D}} \to \mathbf{B}$ be a dg-functor. By Remark 1.3.4, we can find a lift $\widetilde{F}$ of F on objects. Next, we define $\widetilde{F}$ on $\tilde{e}_0, \tilde{e}_1$, applying Lemma 4.3.6 in order to lift $F(\tilde{e}_0)$ and $F(\tilde{e}_1)$ to closed degree 0 morphisms $\widetilde{F}(\tilde{e}_0)$ and $\widetilde{F}(\tilde{e}_1)$. Then, we set $\widetilde{F}(\tilde{e}_1\tilde{e}_0) = \widetilde{F}(\tilde{e}_1)\widetilde{F}(\tilde{e}_0)$; applying again Lemma 4.3.6, we find an element $\widetilde{F}(\tilde{e}_{01})$ which lifts $F(\tilde{e}_{01})$, such that $d\widetilde{F}(\tilde{e}_{01}) = \widetilde{F}(\tilde{e}_1\tilde{e}_0) = \widetilde{F}(d\tilde{e}_{01})$. This clearly suffices to obtain the desired dg-functor $\widetilde{F}$, and hence shows that $\widetilde{\mathbf{D}}$ is cofibrant.

There is a trivial fibration $P\colon \widetilde{\mathbf{D}} \to \mathbf{D}$, defined as follows. P is the identity on objects, moreover:

$$\begin{aligned} G(\tilde{e}_0) &= e_0, \\ G(\tilde{e}_1) &= e_1, \\ G(\tilde{e}_{01}) &= 0. \end{aligned}$$

Now, we are able to show that $\mathbf{D}$ is not cofibrant, as mentioned above. If $\mathbf{D}$ is cofibrant, then $P\colon \widetilde{\mathbf{D}} \to \mathbf{D}$ has a section P':

$$\begin{array}{ccc} & & \widetilde{\mathbf{D}} \\ & \nearrow^{P'} & \downarrow{\scriptstyle P} \\ \mathbf{D} & = & \mathbf{D}. \end{array}$$

P' is the identity on objects, because P is such. Moreover, $P(P'(e_i)) = e_i = P(\tilde{e}_i)$, then $P'(e_i) = \tilde{e}_i$, for $i = 0, 1$. We conclude that $P'(e_1e_0) = \tilde{e}_1\tilde{e}_0 = d\tilde{e}_{01} \neq 0$, which is a contradiction, because $e_1e_0 = 0$ and so $P'(e_1e_0)$ must be 0.

We now come to the counterexample. Let $\mathbf{B}$ be a pretriangulated dg-category, such that $Z^0(\mathbf{B})$ has a zero object, and containing an object not isomorphic to 0 in $H^0(\mathbf{B})$. We define dg-functors $F, G\colon \widetilde{\mathbf{D}} \to \mathbf{B}$ as follows:

$$F(E_i) = G(E_i) = A_i \quad (i = 0, 1, 2)$$

on objects, where $A_0 \ncong 0$ in $H^0(\mathbf{B})$, $A_1 \cong 0$ in $Z^0(\mathbf{B})$, $A_2 = A_0[1]$; moreover, we set

$$\begin{aligned} F(\tilde{e}_i) = G(\tilde{e}_i) &= 0, \quad (i = 0, 1), \\ F(\tilde{e}_{01}) &= 0, \\ G(\tilde{e}_{01}) &= 1_{(A_0,0,1)}. \end{aligned}$$

Clearly, $H^0(F) \cong H^0(G)$, since they are both equal to zero on morphisms. Now, assume that F is homotopic to G. Then, we have a dg-functor $\varphi\colon \widetilde{\mathbf{D}} \to P(\mathbf{B})$ such that $S\varphi = F$ and $T\varphi = G$. In particular:

$$\varphi(E_i) = (A_i, A_i, \lambda_i)$$

on objects, where $\lambda_i\colon A_i \to A_i$ induces an isomorphism in $H^0(\mathbf{B})$ for $i = 0, 1, 2$, and

$$\begin{aligned} \varphi(\tilde{e}_i) &= (0, 0, h_i) \quad (i = 0, 1), \\ \varphi(\tilde{e}_{01}) &= (0, 1_{(A_0,0,1)}, h), \end{aligned}$$

on morphisms, for some suitable maps h_i and h. Notice that $\varphi(\tilde{e}_1)\varphi(\tilde{e}_0) = 0$. The requirement that φ is a dg-functor gives the following conditions:

$$\begin{aligned} dh_i &= 0 \quad (i = 0.1), \\ 0 = \varphi(\tilde{e}_1\tilde{e}_0) &= \varphi(d\tilde{e}_{01}) = d\varphi(\tilde{e}_{01}) = (0, 0, dh + 1_{(A_0,0,1)}\lambda_0). \end{aligned}$$

In particular, we find that $1_{(A_0,0,1)}\lambda_0$ is a coboundary. This in turn implies that λ_0 is a coboundary, since $1_{(A_0,0,1)}$ is closed and invertible. That is, the zero map $A_0 \to A_0$ in $H^0(\mathbf{B})$ is an isomorphism, which means precisely that $A_0 \cong 0$ in $H^0(\mathbf{B})$. Contradiction! We conclude that F and G are not homotopic dg-functors, in spite of being isomorphic when taking H^0.

It is not always possible to lift isomorphisms from the H^0 level

The glueing technique explained in the previous section is based on "partial lifting" results as Lemma 4.3.15. Its failure to give general dg-lift uniqueness results lies in the fact that such partial liftings cannot be extended to global liftings, that is, Lemma 4.3.15 cannot be strengthened, as we are going to see.

Recall that Δ^1 is cofibrant, viewed as a dg-category. So, to define a quasi-functor $F\colon \mathrm{per}_{\mathrm{dg}}(\Delta^1) \to \mathbf{B}$ it is sufficient to define a dg-functor on $\Delta^1 \hookrightarrow \mathrm{per}_{\mathrm{dg}}(\Delta^1)$, if $\mathbf{B}$ is triangulated. Moreover, recall that a natural transformation

$$\overline{\varphi}\colon H^0(F) \to H^0(G)\colon H^0(\mathrm{per}_{\mathrm{dg}}(\Delta^1)) \to H^0(\mathbf{B})$$

is determined by a commutative diagram of the form (4.3.5). That said, we define a quasi-functor $F\colon \mathrm{per}_{\mathrm{dg}}(\Delta^1) \to \mathrm{per}_{\mathrm{dg}}(\Delta^1)$, together with a natural automorphism $\overline{\varphi}\colon H^0(F) \to H^0(F)$, as follows:

$$\begin{array}{ccccccc} E_0 \oplus E_1 & \xrightarrow{\begin{pmatrix} e_0 & 0 \\ 0 & j_0 \end{pmatrix}} & E_1 \oplus \mathrm{C}(e_0) & \xrightarrow{\begin{pmatrix} j_0 & 0 \\ 0 & p_0 \end{pmatrix}} & \mathrm{C}(e_0) \oplus E_0[1] & \xrightarrow{\begin{pmatrix} p_0 & 0 \\ 0 & -e_0[1] \end{pmatrix}} & E_0[1] \oplus E_1[1] \\ \| & & \| & & \downarrow{\scriptstyle\begin{pmatrix} 1 & 0 \\ p_0 & 1 \end{pmatrix}} & & \| \\ E_0 \oplus E_1 & \xrightarrow[\begin{pmatrix} e_0 & 0 \\ 0 & j_0 \end{pmatrix}]{} & E_1 \oplus \mathrm{C}(e_0) & \xrightarrow[\begin{pmatrix} j_0 & 0 \\ 0 & p_0 \end{pmatrix}]{} & \mathrm{C}(e_0) \oplus E_0[1] & \xrightarrow[\begin{pmatrix} p_0 & 0 \\ 0 & -e_0[1] \end{pmatrix}]{} & E_0[1] \oplus E_1[1]. \end{array}$$

With the above commutative diagram, we mean that $F(E_0) = E_0 \oplus E_1, F(E_1) = E_1 \oplus \mathrm{C}(e_0)$, and $F(e_0) = \left(\begin{smallmatrix} e_0 & 0 \\ 0 & j_0 \end{smallmatrix}\right)$ (direct sums in $Z^0(\mathrm{per}_{\mathrm{dg}}(\Delta^1))$); for simplicity, we have identified the dg-functor which defines F with the restriction $F_{|\Delta^1}$. The morphism $\overline{\varphi}$ is defined by the vertical arrows. Notice that $\overline{\varphi}$ differs from the identity just on $\mathrm{C}(e_0)$. Moreover,

$$H^{-1}(\mathrm{Hom}(E_0 \oplus E_1, E_1 \oplus \mathrm{C}(e_0)) \cong 0.$$

The identity morphism $1\colon F \to F$ clearly gives a "partial lift" of $\overline{\varphi}$. Now, we try to find isomorphisms which globally lift $\overline{\varphi}$. We notice the following:

Lemma 4.4.2. *Let* $\mathbf{A}$ *be a dg-category. Let* (A, B, f) *and* (A', B', f') *be objects of* $\underline{\mathrm{Mor}}\,\mathbf{A}$*, and let* $n \in \mathbb{Z}$ *such that*

$$H^{n-1}(\mathbf{A}(A, B')) \cong 0.$$

Next, assume we are given a closed degree n *morphism* $(u, v, h)\colon (A, B, f) \to (A', B', f')$*. Then, if* $u = d\tilde{u}$ *and* $v = d\tilde{v}$*, there exists* $\tilde{h}\colon A \to B'$ *such that*

$$(u, v, h) = d(\tilde{u}, \tilde{v}, \tilde{h}).$$

Proof. By hypothesis we have $d(u, v, h) = 0$, in particular

$$dh + (-1)^n(f'u - vf) = 0.$$

Now, $f'u = d(f'\tilde{u})$ and $vf = d(\tilde{v}f)$, and so

$$d(h + (-1)^n(f'\tilde{u} - \tilde{v}f)) = 0$$

In other words, $h + (-1)^n(f'\tilde{u} - \tilde{v}f)$ is a $(n-1)$-cocycle. Hence, by hypothesis, it is a $(n-1)$-coboundary:

$$h + (-1)^n(f'\tilde{u} - \tilde{v}f) = d\tilde{h}.$$

Finally, we compute:

$$d\begin{pmatrix} \tilde{u} & 0 \\ \tilde{h} & \tilde{v} \end{pmatrix} = \begin{pmatrix} u & 0 \\ h + (-1)^n(f'\tilde{u} - \tilde{v}f) + (-1)^{n-1}(f'\tilde{u} - \tilde{v}f) & v \end{pmatrix} = \begin{pmatrix} u & 0 \\ h & v \end{pmatrix}. \qquad \square$$

Now, let $\varphi\colon F \to F$ be a morphism of quasi-functors such that $H^0(\varphi) = \overline{\varphi}$. We know that φ is completely determined by its restriction $\varphi_{|\Delta^1}\colon F_{|\Delta^1} \to G_{|\Delta^1}$. Recalling Remarks 1.3.16 and 4.2.3, we identify φ to (the class of) a closed degree 0 morphism

$$(\varphi_{E_0}, \varphi_{E_1}, h)\colon (F(E_0), F(E_1), F(e_0)) \to (F(E_0), F(E_1), F(e_0))$$

in $\underline{\mathrm{Mor}}(\mathrm{per}_{\mathrm{dg}}(\Delta^1))$. By hypothesis, $H^0(\varphi_{E_i}) = [1_{E_i}]$ for $i = 0, 1$; so we are in the conditions to apply Lemma 4.4.2 to $(\varphi_{E_0} - 1_{E_0}, \varphi_{E_1} - 1_{E_1}, h)$, and thus deduce that $[(\varphi_{E_0}, \varphi_{E_1}, h)] = [(1_{E_0}, 1_{E_1}, 0)]$. Hence, φ is the identity morphism 1_F, and must induce the identity morphism even of $\mathrm{C}(e_0)$, namely, $\overline{\varphi}_{\mathrm{C}(e_0)} = [1_{\mathrm{C}(e_0)}]$. This is a contradiction, by definition of $\overline{\varphi}$. We have actually proved the following:

Lemma 4.4.3. *Let* $\mathbf{A} = \mathrm{per}_{\mathrm{dg}}(\Delta^1)$*. The functor*

$$\Phi^{\mathbf{A} \to \mathbf{A}}\colon H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{A})) \to \mathrm{Fun}(H^0(\mathbf{A}), H^0(\mathbf{A}))$$

is not full.

A failed attempt and some final considerations

As we have seen in the previous discussion, Lemma 4.3.15 cannot be extended to give "global liftings". Hence, the glueing technique explained in the previous section does not give the dg-lift uniqueness result when the domain dg-category is a glueing of the form

$$\mathbf{A} = \langle E_0, E_1 \rangle \times_C \langle E_2 \rangle,$$

where the abuse of notation $\langle E_0, E_1 \rangle$ here denotes $\mathrm{per}_{\mathrm{dg}}(\Delta^1)$, and C is a direct sum where every object in the set of indecomposables $\{E_0, E_1, \mathrm{C}(e_0)\}$ appears. For example, set $C = (E_0 \oplus E_1 \oplus \mathrm{C}(e_0))[1]$. The resulting dg-category $\mathbf{A}$ is pretriangulated (actually, triangulated) and has a three-term exceptional sequence. With a direct computation, we are able to understand what is the subcategory of generators, namely, the full subcategory of $\mathbf{A}$ whose object set is $\{E_0, E_1, E_2\}$. Indeed, we know that

$$H^0(\mathbf{A})(i_1(-), i_2(E_2)[1]) \cong \langle E_0, E_1 \rangle(-, (E_0 \oplus E_1 \oplus \mathrm{C}(e_0))[1]);$$

so, we find out that the only nontrivial hom-spaces are

$$\begin{aligned} H^0(\mathbf{A})(E_1, E_2) &= \mathbf{k}\langle e_1, e_2 \rangle, \\ H^0(\mathbf{A})(E_0, E_2) &= \mathbf{k}\langle e_2 e_0, e_3 \rangle, \end{aligned}$$

with the only nontrivial relation $e_1 e_0 = 0$. So, $H^0(\mathbf{A})$ is generated by the following diagram:

$$\begin{array}{ccc} E_0 & \xrightarrow{e_0} & E_1 \\ & \underset{e_3}{\searrow} & \Big\downarrow e_1 \ \Big\downarrow e_2 \\ & & E_2. \end{array}$$

Now, let $\mathbf{E}$ be a cofibrant replacement of the $\mathbf{k}$-linear category described by the above picture: to obtain it, just change $e_1 e_0 = 0$ to $e_1 e_0 = d e_{01}$, in a similar fashion as (4.4.1).

The dg-category $\mathrm{pretr}(\mathbf{E})$ is apparently a good candidate for finding a counterexample to dg-lift uniqueness. The idea is to define dg-functors $F, G \colon \mathbf{E} \to \mathbf{B}$, with $\mathbf{B}$ a suitable pretriangulated dg-category, show that they are not homotopic but their extensions $\widetilde{F}, \widetilde{G} \colon \mathrm{pretr}(\mathbf{E}) \to \mathbf{B}$ are isomorphic in cohomology: $H^0(\widetilde{F}) \cong H^0(\widetilde{G})$. Showing that F is not homotopic to G and that $H^0(F) \cong H^0(G)$ is a rather simple task; unfortunately, finding an isomorphism in cohomology of their extensions has proved to be a very hard problem, for which we ran out of ideas. Nevertheless, we sketch our attempt in the following example.

Example 4.4.4. We start by defining dg-functors $F, G \colon \mathbf{E} \to \mathrm{pretr}(\Delta^1)$. View $\mathbf{E}$ as a full subcategory of $\mathrm{pretr}(\mathbf{E})$. Let $j_0 \colon E_1 \to \mathrm{C}(e_0)$ and $i_0 \colon E_0 \to \mathrm{C}(e_0)$ be the inclusion maps associated to $\mathrm{C}(e_0)$: j_0 is a closed degree 0 maps, whereas i_0 has degree -1 and $d i_0 = j_0 e_0$. Since $e_1 e_0 = d e_{01}$, we may apply the universal property of the cone and find out that there exists a unique closed degree 0 map $e_1' \colon \mathrm{C}(e_0) \to E_2$ such that

$$\begin{aligned} e_1 &= e_1' j_0, \\ e_{01} &= e_1' i_0. \end{aligned}$$

So, we find out that in order to define a dg-functor F on $\mathbf{E}$, it is sufficient to define it on objects $E_0, E_1, E_2, \mathrm{C}(e_0)$, in such a way that $F(\mathrm{C}(e_0)) \cong \mathrm{C}(F(e_0))$, and on the closed degree 0 morphisms e_0, e_1', e_2, e_3. Indeed, $F(j_0)$ and $F(i_0)$ are forced to be the canonical maps associated to the cone $\mathrm{C}(F(e_0))$, and the values $F(e_1)$ and $F(e_{01})$ are obtained from the above relations. That said, we define $F, G \colon \mathbf{E} \to \operatorname{pretr}(\Delta^1)$ as follows:

$$\begin{aligned} F(E_0) &= G(E_0) = E_0 \oplus E_1, \\ F(E_1) &= G(E_1) = E_1 \oplus \mathrm{C}(e_0), \\ F(\mathrm{C}(e_0)) &= G(\mathrm{C}(e_0)) = \mathrm{C}(e_0) \oplus E_0[1], \\ F(E_2) &= G(E_2) = \mathrm{C}(e_0) \oplus E_0[1] \oplus E_1, \end{aligned}$$

$$\begin{aligned} F(e_0) &= G(e_0) = \begin{pmatrix} e_0 & 0 \\ 0 & j_0 \end{pmatrix}, \\ F(e_1') &= G(e_1') = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ F(e_2) &= \begin{pmatrix} j_0 & -1 \\ 0 & p_0 \\ 0 & 0 \end{pmatrix}, \\ G(e_2) &= \begin{pmatrix} j_0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ F(e_3) &= G(e_3) = \begin{pmatrix} 0 & j_0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Direct sums are taken in $Z^0(\operatorname{per_{dg}}(\Delta^1))$. We identified $E_0[1] = \mathrm{C}(j_0)$ in the definitions of $F(\mathrm{C}(e_0))$ and $F(E_2)$, which is actually an abuse of notation; analogously, the canonical inclusion map $\mathrm{C}(e_0) \to \mathrm{C}(j_0)$ is identified with the projection $p_0 \colon \mathrm{C}(e_0) \to E_0[1]$. With this identification, we have that

$$\begin{aligned} F(j_0) &= \begin{pmatrix} j_0 & 0 \\ 0 & p_0 \end{pmatrix}, \\ F(p_0) &= \begin{pmatrix} p_0 & 0 \\ 0 & -e_0[1] \end{pmatrix}. \end{aligned}$$

It can be shown that F and G are not homotopic. To see this, it is useful to remark that a homotopy $\varphi \colon F \to G$ is given by homotopy equivalences $\varphi_{E_i} \colon F(E_i) \to G(E_i)$ for all $i = 0, 1, 2$ subject to the various compatibilities in H^0, together with the following commutative diagram in H^0:

$$\begin{array}{ccc} F(\mathrm{C}(e_0)) & \xrightarrow{\varphi_{\mathrm{C}(e_0)}} & G(\mathrm{C}(e_0)) \\ \Big\downarrow{\scriptstyle F(e_1')} & & \Big\downarrow{\scriptstyle G(e_1')} \\ F(E_2) & \xrightarrow{\varphi_{E_2}} & G(E_2), \end{array}$$

where $\varphi_{\mathrm{C}(e_0)}$ is "functorial": it is the cone of $\varphi(e_0) \colon \varphi_{E_0} \to \varphi_{E_1}$ in $H^0(\underline{\operatorname{Mor}}(\operatorname{per_{dg}}(\Delta^1)))$ (recall that φ is a dg-functor from $\mathbf{E}$ to the dg-category of morphisms). The details are left to the reader.

Next, we attempted to show that the extensions $\widetilde{F}, \widetilde{G} \colon \mathrm{pretr}(\mathbf{E}) \to \mathrm{per}_{\mathrm{dg}}(\Delta^1)$ were isomorphic in cohomology, but this is actually false, as we are going to show. Both the domain and codomain dg-categories are smooth (Proposition 4.3.12) and locally perfect, so by Theorem 3.5.9 we know that quasi-functors $\mathrm{pretr}(\mathbf{E}) \to \mathrm{per}_{\mathrm{dg}}(\Delta^1)$ and $\mathrm{per}_{\mathrm{dg}}(\Delta^1) \to \mathrm{pretr}(\mathbf{E})$ have adjoints. By Theorem 4.3.8, we know that $\Phi^{\mathrm{per}_{\mathrm{dg}}(\Delta^1) \to \mathrm{pretr}(\mathbf{E})}$ is essentially injective; by Proposition 3.6.3, we know that this implies (it is actually equivalent to) the essential injectivity of $\Phi^{\mathrm{pretr}(\mathbf{E}) \to \mathrm{per}_{\mathrm{dg}}(\Delta^1)}$. Hence, by the fact that F is not homotopic to G, we deduce that $H^0(\widetilde{F}) \not\cong H^0(\widetilde{G})$. In spite of this, it can be shown that there exist homotopy equivalences

$$\begin{aligned} \varphi_{E_0} &\colon F(E_0) \to G(E_0), \\ \varphi_{E_1} &\colon F(E_1) \to G(E_1), \\ \varphi_{\mathrm{C}(e_0)} &\colon F(\mathrm{C}(e_0) \to G(\mathrm{C}(e_0)), \\ \varphi_{E_2} &\colon F(E_2) \to G(E_2) \end{aligned}$$

such that the following diagram, which is the analogue of (4.3.4), is commutative in $H^0(\mathrm{per}_{\mathrm{dg}}(\Delta^1))$:

$$\begin{array}{ccc} F(E_0 \oplus E_1 \oplus \mathrm{C}(e_0)) & \xrightarrow{H^0(F)(\alpha)} & F(E_2) \\ {\scriptstyle \varphi_{E_0} \oplus \varphi_{E_1} \oplus \varphi_{\mathrm{C}(e_0)}} \downarrow & & \downarrow {\scriptstyle \varphi_2} \\ G(E_0 \oplus E_1 \oplus \mathrm{C}(e_0)) & \xrightarrow{H^0(G)(\alpha)} & G(E_2). \end{array}$$

A direct computation (left as an exercise) shows that the morphism α is given by

$$\alpha = \begin{pmatrix} e_3, e_2, e_1' \end{pmatrix}.$$

The above homotopy equivalences can be defined as follows:

$$\begin{aligned} \varphi_{E_0} &= 1_{F(E_0)}, \\ \varphi_{E_1} &= 1_{F(E_1)}, \\ \varphi_{\mathrm{C}(e_0)} &= \begin{pmatrix} 1 & 0 \\ p_0 & 1 \end{pmatrix}, \\ \varphi_{E_2} &= \begin{pmatrix} 1 & 0 & 0 \\ p_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

So, this example, even if ultimately unsatisfactory, shows that the glueing technique *needs* the condition that the morphism $Fi_1N[-1] \to Gi_1N[-1]$ is induced by a *morphism of quasi-functors* $Fi_1 \to Gi_1$.

The above example can be modified taking $\mathrm{pretr}(\mathbf{E})$ as the target dg-category, in the hope of finding a counterexample. Unfortunately, this dg-category is complicated enough to make the task of defining an isomorphism of functors in H^0 very hard (for example, we are unable to describe the indecomposable objects of $\mathrm{tria}(\mathbf{E})$). The author is even unable to conjecture whether the dg-lift uniqueness should hold true or not, for quasi-functors defined on this dg-category.

Chapter 5

A_∞-functors

Another way to study the homotopy theory of dg-categories comes from the theory of A_∞-*categories* and A_∞-functors. They are, respectively, a homotopy coherent incarnation of dg-categories and dg-functors. A_∞-functors are actually a model for quasi-functors; their advantage over quasi-representable bimodules relies in their "concreteness": they are defined by elementary (even if quite complicated) formulae, which can be employed in rather direct arguments. This formalism will allow us to prove a dg-lift uniqueness result with some additional hypothesis on the functors involved; the result has some interesting geometric applications.

5.1 A_∞-categories and functors

From now on, $\mathbf{k}$ will be assumed to be a field. The basic notions of the theory of A_∞-categories and functors are taken directly from [Sei08], whose conventions will be followed. We warn the reader especially about sign conventions, which are possibly the most annoying feature of the theory. If it feels more comfortable, just assume that $\operatorname{char}\mathbf{k} = 2$, at least at a first reading.

We will be working with *strictly unital* A_∞-categories and functors. The formal definitions are as follows:

Definition 5.1.1. A *strictly unital* A_∞-category $\mathbf{A}$ consists of a set of objects $\operatorname{Ob}\mathbf{A}$, a graded $\mathbf{k}$-vector space $\mathbf{A}(X_0, X_1)$ for any couple of objects $X_0, X_1 \in \mathbf{A}$, and multilinear composition maps for any order $d \geq 1$:

$$\mu^d_{\mathbf{A}} \colon \mathbf{A}(X_{d-1}, X_d) \otimes \ldots \otimes \mathbf{A}(X_0, X_1) \to \mathbf{A}(X_0, X_d)[2-d], \tag{5.1.1}$$

satisfying the following collection of equations (for all $d \geq 1$):

$$\sum_{m=1}^{d} \sum_{n=0}^{d-m} (-1)^{\maltese_n} \mu^{d-m+1}_{\mathbf{A}}(f_d, \ldots, f_{n+m+1}, \mu^m_{\mathbf{A}}(f_{n+m}, \ldots, f_{n+1}), f_n, \ldots, f_1) = 0, \tag{5.1.2}$$

where by definition $\maltese_n = |f_1| + \ldots + |f_n| - n$. Moreover, for any object $X \in \mathbf{A}$, there exists a (necessarily unique) morphism $1_X \in \mathbf{A}(X,X)^0$ which satisfies:

$$\begin{aligned}
&\mu^1_{\mathbf{A}}(1_X) = 0,\\
&(-1)^{|f|}\mu^2_{\mathbf{A}}(1_{X_1}, f) = \mu^2_{\mathbf{A}}(f, 1_{X_0}) = f, \quad \forall\, f \in \mathbf{A}(X_0, X_1),\\
&\mu^d_{\mathbf{A}}(f_{d-1}, \ldots, f_{n+1}, 1_{X_n}, f_n, \ldots, f_1) = 0,\\
&\forall\, d > 2, f_k \in \mathbf{A}(X_{k-1}, X_k), \forall\, 0 \leq n < d.
\end{aligned} \tag{5.1.3}$$

Unwinding the above definition, we find out that that the map $\mu^1_{\mathbf{A}}$ is a coboundary which endows the hom-spaces $\mathbf{A}(X,Y)$ with a structure of chain complex. The composition $\mu^2_{\mathbf{A}}$ is not associative, but its deviation from being so is measured by the higher order maps $\mu^d_{\mathbf{A}}$.

Definition 5.1.2. Let $\mathbf{A}$ and $\mathbf{B}$ be (strictly unital) A_∞-categories. An A_∞-functor $F\colon \mathbf{A} \to \mathbf{B}$ consists of a map of sets

$$\begin{aligned}
F^0\colon\ &\operatorname{Ob}\mathbf{A} \to \operatorname{Ob}\mathbf{B},\\
&X \mapsto F^0(X) = F(X),
\end{aligned}$$

and multilinear maps

$$F^d\colon \mathbf{A}(X_{d-1}, X_d) \otimes \ldots \otimes \mathbf{A}(X_0, X_1) \to \mathbf{B}(F(X_0), F(X_d))[1-d], \tag{5.1.4}$$

subject to the following equations, for all $d \geq 1$:

$$\begin{aligned}
&\sum_{r\geq 1}\sum_{s_1+\ldots+s_r=d} \mu^r_{\mathbf{B}}(F^{s_r}(f_d, \ldots, f_{d-s_r+1}), \ldots, F^{s_1}(f_{s_1}, \ldots, f_1))\\
&\quad = \sum_{m=1}^{d}\sum_{n=0}^{d-m}(-1)^{\maltese_n} F^{d-m+1}(f_d, \ldots, f_{n+m+1}, \mu^m_{\mathbf{A}}(f_{n+m}, \ldots, f_{n+1}), f_n, \ldots, f_1),
\end{aligned} \tag{5.1.5}$$

where $s_i \geq 1$ for all i. Moreover, F is required to satisfy the following strict unitality condition:

$$\begin{aligned}
&F^1(1_X) = 1_{F(X)}, \quad \forall\, X \in \mathbf{A},\\
&F^d(f_{d-1}, \ldots, f_{n+1}, 1_{X_n}, f_n, \ldots, f_1) = 0,\\
&\forall\, d \geq 2, f_k \in \mathbf{A}(X_{k-1}, X_k), \forall\, 0 \leq n < d.
\end{aligned} \tag{5.1.6}$$

Given A_∞-functors $F\colon \mathbf{A} \to \mathbf{B}$ and $G\colon \mathbf{B} \to \mathbf{C}$, their *composition* $G \circ F$ is defined as follows:

$$\begin{aligned}
&(G \circ F)^0 = G^0 \circ F^0,\\
&(G \circ F)^d(f_d, \ldots, f_1)\\
&\qquad = \sum_{r\geq 1}\sum_{s_1+\ldots+s_r=d} G^r(F^{s_r}(f_d, \ldots, f_{d-s_r+1}), \ldots, F^{s_1}(f_{s_1}, \ldots, f_1)),
\end{aligned} \tag{5.1.7}$$

whenever $d \geq 1$, with $s_i \geq 1$.

Remark 5.1.3. Any dg-category $\mathbf{A}$ can be viewed as an A_∞-category, setting

$$\begin{aligned} \mu^1_{\mathbf{A}}(f) &= (-1)^{|f|} df, \\ \mu^2_{\mathbf{A}}(g,f) &= (-1)^{|f|} gf, \\ \mu^d_{\mathbf{A}} &= 0, \quad \forall\, d > 2. \end{aligned}$$

As we see, apart from sign twists, a dg-category is an A_∞-category whose higher compositions (for $d > 2$) vanish. From now on, any dg-category will be implicitly viewed in this way as an A_∞-category.

It is interesting to see how the definition of A_∞-functor behaves if the domain and codomain are assumed to be dg-categories. If $F\colon \mathbf{A} \to \mathbf{B}$ is an A_∞-functor between dg-categories, the degree d equation (5.1.5) boils down to:

$$\begin{aligned} &\mu^1_{\mathbf{B}}(F^d(f_d,\dots,f_1)) + \sum_{j=1}^{d-1} \mu^2_{\mathbf{B}}(F^j(f_d,\dots,f_{d-j+1}), F^{d-j}(f_{d-j},\dots,f_1)) \\ &= \sum_{n=0}^{d-1} (-1)^{\maltese_n} F^d(f_d,\dots,f_{n+2},\mu^1_{\mathbf{A}}(f_{n+1}),f_n,\dots,f_1) \\ &\qquad + \sum_{n=0}^{d-2} (-1)^{\maltese_n} F^{d-1}(f_d,\dots,f_{n+3},\mu^2_{\mathbf{A}}(f_{n+2},f_{n+1}),f_n,\dots,f_1). \end{aligned} \tag{5.1.8}$$

In the even simpler case when $F\colon \mathbb{E} \to \mathbf{B}$ is an A_∞-functor where $\mathbb{E}$ is a $\mathbf{k}$-linear category and $\mathbf{B}$ is a dg-category, the degree d equation defining F then reduces to the following:

$$\begin{aligned} &\mu^1_{\mathbf{B}}(F^d(f_d,\dots,f_1)) + \sum_{j=1}^{d-1} \mu^2_{\mathbf{B}}(F^j(f_d,\dots,f_{d-j+1}), F^{d-j}(f_{d-j},\dots,f_1)) \\ &\qquad = \sum_{n=0}^{d-2} (-1)^{\maltese_n} F^{d-1}(f_d,\dots f_{n+3},\mu^2_{\mathbb{E}}(f_{n+2},f_{n+1}),f_n,\dots,f_1). \end{aligned} \tag{5.1.9}$$

It is also interesting to see what is the composition of an A_∞-functor $F\colon \mathbf{A} \to \mathbf{B}$ (between dg-categories) with a dg-functor $G\colon \mathbf{B} \to \mathbf{C}$. Such a dg-functor, viewed as an A_∞-functor, is characterised by having $G^d = 0$ for all $d > 1$. Formula (5.1.7) becomes very simple:

$$(G \circ F)^d(f_d,\dots,f_1) = G^1(F^d(f_d,\dots,f_1)), \tag{5.1.10}$$

for all $d \geq 1$.

Example 5.1.4. Let $\mathbf{A}$ be a dg-category, viewed as an A_∞-category. As an exercise, let us write down what happens when we view the dg-category of morphisms $\mathbf{Q} = \underline{\operatorname{Mor}}\,\mathbf{A}$

(see Definition 1.3.11) as an A_∞-category. First:

$$\begin{aligned}\mu^1_{\mathbf{Q}}\begin{pmatrix} u & 0 \\ h & v\end{pmatrix} &= (-1)^{|u|}\begin{pmatrix} du & 0 \\ dh + (-1)^{|u|}(f'u - vf) & dv\end{pmatrix} \\ &= (-1)^{|u|}\begin{pmatrix} (-1)^{|u|}\mu^1_{\mathbf{A}}(u) & 0 \\ (-1)^{|u|-1}\mu^1_{\mathbf{A}}(h) + (-1)^{|u|}((-1)^{|u|}\mu^2_{\mathbf{A}}(f',u) - \mu^2_{\mathbf{A}}(v,f)) & (-1)^{|u|}\mu^1_{\mathbf{A}}(v)\end{pmatrix} \\ &= \begin{pmatrix} \mu^1_{\mathbf{A}}(u) & 0 \\ -\mu^1_{\mathbf{A}}(h) + (-1)^{|u|}\mu^2_{\mathbf{A}}(f',u) - \mu^2_{\mathbf{A}}(v,f) & \mu^1_{\mathbf{A}}(v)\end{pmatrix}.\end{aligned}$$

Moreover:

$$\begin{aligned}&\mu^2_{\mathbf{Q}}\left(\begin{pmatrix} u' & 0 \\ h' & v'\end{pmatrix}, \begin{pmatrix} u & 0 \\ h & v\end{pmatrix}\right) \\ &= \begin{pmatrix} (-1)^{|u|}u'u & 0 \\ (-1)^{|u|}((-1)^{|u|}h'u + v'h) & (-1)^{|u|}v'v\end{pmatrix} \\ &= \begin{pmatrix} \mu^2_{\mathbf{A}}(u',u) & 0 \\ (-1)^{|u|}\mu^2_{\mathbf{A}}(h',u) - \mu^2_{\mathbf{A}}(v',h) & \mu^2_{\mathbf{A}}(v',v)\end{pmatrix}.\end{aligned}$$

Given A_∞-categories $\mathbf{A}$ and $\mathbf{B}$, there is an *A_∞-category* $\mathrm{Fun}_\infty(\mathbf{A},\mathbf{B})$ *of (strictly unital) A_∞-functors*. Its definition involves describing *(A_∞-)natural transformations* of A_∞-functors.

Definition 5.1.5. Let $F, G\colon \mathbf{A} \to \mathbf{B}$ be A_∞-functors. A *degree g pre-natural transformation $h\colon F \to G$* is consists of a sequence of maps $(h^0, h^1, \dots)$ such that

$$h^0\colon X \mapsto h^0_X \in \mathbf{B}(F(X), G(X))^g, \quad X \in \mathbf{A},$$

and h^d is a family of multilinear maps

$$h^d\colon \mathbf{A}(X_{d-1}, X_d) \otimes \ldots \otimes \mathbf{A}(X_0, X_1) \to \mathbf{B}(F(X_0), G(X_d))[g-d]$$

for any family of objects $X_0, \dots, X_d \in \mathbf{A}$. Pre-natural transformations $F \to G$ form the graded vector space $\mathrm{Fun}_\infty(\mathbf{A},\mathbf{B})(F,G)$. Compositions are described in [Sei08, Paragraph (1d)]. For example, we have that

$$\mu^1(h)^0_X = \mu^1_{\mathbf{B}}(h^0_X), \quad \forall\, X \in \mathbf{A}.$$

Moreover, we require the strict unitality condition:

$$h^d(f_{d-1}, \dots, f_{n+1}, 1_{X_n}, f_n, \dots, f_1) = 0, \tag{5.1.11}$$

for all $d \geq 1$ and $0 \leq n < d$, with $f_k \in \mathbf{A}(X_{k-1}, X_k)$.

Remark 5.1.6. It is worth writing down the coboundary formula for a pre-natural transformation $h\colon F \to G$ when $F, G\colon \mathbf{A} \to \mathbf{B}$ are A_∞-functors between dg-categories. If $d \geq 1$, we have:

$$\mu^1(h)^d(f_d, \dots, f_1) = A^d - B^d, \tag{5.1.12}$$

where

$$\begin{aligned} A^d = &\mu^1_{\mathbf{B}}(h^d(f_d, \dots, f_1)) \\ &+ \mu^2_{\mathbf{B}}(G^d(f_d, \dots, f_1), h_{X_0}) + (-1)^{\maltese_d(|h|-1)} \mu^2_{\mathbf{B}}(h_{X_d}, F^d(f_d, \dots, f_1)) \\ &+ \sum_{j=1}^{d-1} \mu^2_{\mathbf{B}}(G^j(f_d, \dots, f_{d-j+1}), h^{d-j}(f_{d-j}, \dots, f_1)) \\ &+ \sum_{j=1}^{d-1} (-1)^{\maltese_{d-j}(|h|-1)} \mu^2_{\mathbf{B}}(h^j(f_d, \dots, f_{d-j+1}), F^{d-k}(f_{d-j}, \dots, f_1)), \end{aligned} \tag{5.1.13}$$

and

$$\begin{aligned} B^d = &\sum_{n=0}^{d-1} (-1)^{\maltese_n + |h| - 1} h^d(f_d, \dots, f_{n+2}, \mu^1_{\mathbf{A}}(f_{n+1}), f_n, \dots, f_1) \\ &+ \sum_{n=0}^{d-2} (-1)^{\maltese_n + |h| - 1} h^{d-1}(f_d, \dots, f_{n+3}, \mu^2_{\mathbf{A}}(f_{n+2}, f_{n+1}), f_n, \dots, f_1), \end{aligned} \tag{5.1.14}$$

given composable morphisms $f_1, \dots, f_d$ with first source X_0 and final target X_d. Notice that the term B_d is similar to the right hand side of (5.1.8).

Closed degree 0 pre-natural transformations of A_∞-functors are properly called *natural transformations*. They can be characterised as directed homotopies, in the sense explained by the following result.

Lemma 5.1.7. *Let $\mathbf{A}, \mathbf{B}$ be dg-categories. Let $F, G\colon \mathbf{A} \to \mathbf{B}$ be A_∞-functors. There is a bijection between the set of (closed, degree 0) natural transformations $F \to G$ and the set of A_∞-functors $\varphi\colon \mathbf{A} \to \underline{\mathrm{Mor}}\,\mathbf{B}$ such that $S\varphi = F$ and $T\varphi = G$:*

$$\varphi^d = (F^d, G^d, h^d) \leftrightarrow h^d. \tag{5.1.15}$$

Proof. Let $\varphi\colon \mathbf{A} \to \underline{\mathrm{Mor}}\,\mathbf{B}$ an A_∞-functor as in the hypothesis. In particular, for any string of composable maps $f_1, \dots, f_d$ with first source X_0 and final target X_d, we have

$$\varphi^d(f_d, \dots, f_1) = (F^d(f_d, \dots, f_1), G^d(f_d, \dots, f_1), h^d(f_d, \dots, f_1))$$

as a morphism $(F(X_0), G(X_0), h_{X_0}) \to (F(X_d), G(X_d), h_{X_d})$. Notice that $F^d(\dots)$ and $G^d(\dots)$ have degree $|f_1| + \dots + |f_d| + 1 - d$, that is, $\maltese_d + 1$, whereas $h^d(\dots)$ has degree $\maltese_d$. Now, we unwind the equation (5.1.8) which defines φ. By Example 5.1.4, we have

$$\mu^1(\varphi^d) = (\mu^1_{\mathbf{B}}(F^d), \mu^1_{\mathbf{B}}(G^d), -\mu^1_{\mathbf{B}}(h^d) + (-1)^{\maltese_d + 1} \mu^2_{\mathbf{B}}(h_{X_d}, F^d) - \mu^2_{\mathbf{B}}(G^d, h_{X_0})).$$

Moreover:

$$\begin{aligned}
&\mu^2(\varphi^j(f_d,\dots,f_{d-j+1}),\varphi^{d-j}(f_{d-j},\dots,f_1))\\
&\quad=\mu^2((F^j,G^j,h^j),(F^{d-j},G^{d-j},h^{d-j}))\\
&\quad=(\mu^2_{\mathbf{B}}(F^j,F^{d-j}),\mu^2_{\mathbf{B}}(G^j,G^{d-j}),(-1)^{\maltese_{d-j}+1}\mu^2_{\mathbf{B}}(h^j,F^{d-j})-\mu^2_{\mathbf{B}}(G^j,h^{d-j})).
\end{aligned}$$

Now, we find out that the left hand side of (5.1.8), projected to the third component, is equal to the following:

$$\begin{aligned}
&-\mu^1_{\mathbf{B}}(h^d(f_d,\dots,f_1))-\mu^2_{\mathbf{B}}(G^d(f_d,\dots,f_1),h_{X_0})-(-1)^{\maltese_d}\mu^2_{\mathbf{B}}(h_{X_d},F^d(f_d,\dots,f_1))\\
&\qquad-\sum_{j=1}^{d-1}\mu^2_{\mathbf{B}}(G^j(f_d,\dots,f_{d-j+1}),h^{d-j}(f_{d-j},\dots,f_1))\\
&\qquad-\sum_{j=1}^{d-1}(-1)^{\maltese_{d-j}}\mu^2_{\mathbf{B}}(h^j(f_d,\dots,f_{d-j+1}),F^{d-j}(f_{d-j},\dots,f_1)).
\end{aligned}$$

We immediately notice that the above term is equal to $-A^d$ when $|h|=0$ (see (5.1.13)). Moreover, the right hand side of (5.1.8), projected to the third component, is equal to $-B^d$ when $|h|=0$ (see (5.1.14)). Now, it is clear that any A_∞-functor $\varphi\colon \mathbf{A}\to \underline{\mathrm{Mor}}\,\mathbf{B}$ such that $S\varphi=F$ and $T\varphi=G$ defines a closed degree 0 natural transformation $h\colon F\to G$, taking the projection of φ to the third component; conversely, given $h\colon F\to G$ closed and of degree 0, setting

$$\varphi^d=(F^d,G^d,h^d)$$

we obtain an A_∞-functor with the desired properties. Clearly, these mappings are mutually inverse. Moreover, the scrict unitality condition (5.1.6) for φ is clearly equivalent to the strict unitality condition (5.1.11) for h. □

If $\mathbf{A}$ and $\mathbf{B}$ are dg-categories, then so is $\mathrm{Fun}_\infty(\mathbf{A},\mathbf{B})$. Actually, this is an incarnation of the internal hom in Hqe, as mentioned in [Kel06, Paragraph 4.3]:

Proposition 5.1.8. *The dg-category $\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A},\mathbf{B})$ can be identified with the dg- category $\mathrm{Fun}_\infty(\mathbf{A},\mathbf{B})$ of strictly unital A_∞-functors from $\mathbf{A}$ to $\mathbf{B}$.*

The functor $\Phi^{\mathbf{A}\to\mathbf{B}}$ has clearly an incarnation in this setting:

$$\begin{aligned}
\Phi^{\mathbf{A}\to\mathbf{B}}\colon\ & H^0(\mathrm{Fun}_\infty(\mathbf{A},\mathbf{B}))\to\mathrm{Fun}(H^0(\mathbf{A}),H^0(\mathbf{B})),\\
& F\mapsto H^0(F),\quad H^0(F)(f)=[F^1(f)],\\
& [h]_{\mu^1}\mapsto H^0(h),\quad H^0(h)_X=[h^0_X],
\end{aligned}\tag{5.1.16}$$

where here $[\cdot]_{\mu^1}$ denotes the cohomology class with respect of the coboundary μ^1 of $\mathrm{Fun}_\infty(\mathbf{A},\mathbf{B})(F,G)$. Recalling Lemma 5.1.7, the action of the above functor on morphisms can also be viewed in terms of directed homotopies. Given $\varphi\colon \mathbf{A}\to\underline{\mathrm{Mor}}\,\mathbf{B}$ such that $S\varphi=F$ and $T\varphi=G$, we may identify $H^0(\varphi)$ to the ordinary functor

$$H^0(\mathbf{A})\to\mathrm{Mor}(H^0(\mathbf{B}))$$

obtained by the following composition:

$$H^0(\mathbf{A}) \to H^0(\underline{\mathrm{Mor}}\,\mathbf{B}) \xrightarrow{(1.3.6)} \mathrm{Mor}(H^0(\mathbf{B})).$$

5.2 Uniqueness of dg-lifts

The goal of this section is to prove a dg-lift uniqueness result using the formalism and techniques of A_∞-functors. We will need the following (simplified) obstruction theory result, which can be proved with a direct computation. The analogue (general) result is proved for A_∞-algebras in [LH03, Corollaire B.1.5].

Lemma 5.2.1. *Let $\mathbb{E}$ be a $\mathbf{k}$-linear category, let $\mathbf{B}$ be a dg-category, and let $n \geq 2$ be an integer. Suppose that we have a finite sequence $(F^0, F^1, \dots, F^{n-1})$, where $F^0 \colon \mathrm{Ob}\,\mathbb{E} \to \mathrm{Ob}\,\mathbf{B}, X \mapsto F(X) = F^0(X)$ and*

$$F^d \colon \mathbb{E}(X_{d-1}, X_d) \otimes \ldots \otimes \mathbb{E}(X_0, X_1) \to \mathbf{B}(F(X_0), F(X_d))[1-d],$$

is a multilinear map, for all $d = 1, \dots, n-1$. Assume that (5.1.9) is satisfied for all $d = 1, \dots, n-1$. Then, the expression

$$\sum_{j=0}^{n-2} (-1)^{\maltese_j} F^{n-1}(f_n, \dots f_{j+3}, \mu^2_{\mathbb{E}}(f_{j+2}, f_{j+1}), f_j, \dots, f_1) \\ - \sum_{j=1}^{n-1} \mu^2_{\mathbf{B}}(F^j(f_n, \dots, f_{n-j+1}), F^{n-j}(f_{n-j}, \dots, f_1))$$

is a $\mu^1_{\mathbf{B}}$-cocycle, for any chain of composable maps $f_1, \dots, f_n$.

Another key tool in our argument is Lemma 4.4.2, which we write down in its "A_∞ version":

Lemma 5.2.2. *Let $\mathbf{A}$ be a dg-category. Let (A, B, f) and (A', B', f') be objects in $\mathbf{Q} = \underline{\mathrm{Mor}}\,\mathbf{A}$ (viewed as an A_∞-category), and let $n \in \mathbb{Z}$ such that*

$$H^{n-1}(\mathbf{A}(A, B')) = 0.$$

Next, assume that we are given a degree n morphism $(u, v, h) \colon (A, B, f) \to (A', B', f')$ such that $\mu^1_{\mathbf{Q}}(u, v, h) = 0$. Then, if $u = \mu^1_{\mathbf{A}}(\tilde{u})$ and $v = \mu^1_{\mathbf{A}}(\tilde{v})$, there exists $\tilde{h} \colon A \to B'$ such that

$$(u, v, h) = \mu^1_{\mathbf{Q}}(\tilde{u}, \tilde{v}, \tilde{h}).$$

Proof. Recall Example 5.1.4. $u = \mu^1_{\mathbf{A}}(\tilde{u})$ means $(-1)^{n-1}u = d\tilde{u}$, and analogously $(-1)^{n-1}v = d\tilde{v}$. Apply Lemma 4.4.2 to $(-1)^{n-1}(u, v, h)$:

$$(-1)^{n-1}(u, v, h) = d(\tilde{u}, \tilde{v}, \tilde{h}) = (-1)^{n-1}\mu^1_{\mathbf{Q}}(\tilde{u}, \tilde{v}, \tilde{h}),$$

and the claim follows. □

We are going to prove the following claim, which is actually a lifting result of natural transformations:

Proposition 5.2.3. *Let $\mathbb{E}$ be a $\mathbf{k}$-linear category, viewed as a dg-category concentrated in degree 0, and let $\mathbf{B}$ be a dg-category. Let $F, G\colon \mathbb{E} \to \mathbf{B}$ be quasi-functors, such that*

$$H^j(\mathbf{B}(F(E), G(E'))) = 0, \tag{5.2.1}$$

for all $j < 0$ and for all $E, E' \in \mathbb{E}$. Let $\overline{\varphi}\colon H^0(F) \to H^0(G)$ be a natural transformation. Then, there exists a morphism $\varphi\colon F \to G$ in $H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbb{E}, \mathbf{B}))$ such that $H^0(\varphi) = \overline{\varphi}$.

We obtain the following theorem, which is the announced dg-lift uniqueness result:

Theorem 5.2.4. *Let $\mathbb{E}$ be a $\mathbf{k}$-linear category, viewed as a dg-category concentrated in degree 0, and let $\mathbf{B}$ be a triangulated dg-category. Let $F, G\colon \mathbb{E} \to \mathbf{B}$ be quasi-functors, such that*

$$H^j(\mathbf{B}(F(E), F(E'))) \cong 0, \tag{5.2.2}$$

for all $j < 0$, for all $E, E' \in \mathbb{E}$. Let $\overline{\varphi}\colon H^0(F) \to H^0(G)$ be a natural isomorphism. Then, there exists an isomorphism $\varphi\colon F \to G$ in $H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbb{E}, \mathbf{B}))$ such that $H^0(\varphi) = \overline{\varphi}$.

In particular, set $\mathbf{A} = \mathrm{per}_{\mathrm{dg}}(\mathbb{E})$; if $F, G\colon \mathbf{A} \to \mathbf{B}$ are quasi-functors satisfying (5.2.2), then $H^0(F) \cong H^0(G)$ implies $F \cong G$ in $H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B}))$.

Proof. Since $H^0(F) \cong H^0(G)$ and $\mathbf{B}$ is triangulated, then (5.2.1) holds. Then, the proof is a direct application of Proposition 5.2.3, Proposition 3.6.7. The second part of the statement follows from Lemma 3.6.5. □

Upon identifying $\mathbb{R}\underline{\mathrm{Hom}}(\mathbb{E}, \mathbf{B})$ to $\mathrm{Fun}_\infty(\mathbb{E}, \mathbf{B})$, Proposition 5.2.3 is translated to the following:

Proposition 5.2.5. *Let $\mathbb{E}$ be a $\mathbf{k}$-linear category, viewed as a dg-category concentrated in degree 0, and let $\mathbf{B}$ be a dg-category. Let $F, G\colon \mathbb{E} \to \mathbf{B}$ be (strictly unital) A_∞-functors, satisfying*

$$H^j(\mathbf{B}(F^0(E), G^0(E'))) = 0, \tag{5.2.3}$$

for all $j < 0$, for all $E, E' \in \mathbb{E}$. Assume $\overline{\varphi}\colon H^0(F) \to H^0(G)$ is a natural transformation. Then, there exists an A_∞-natural transformation $\varphi\colon F \to G$, such that $H^0(\varphi) = \overline{\varphi}$.

Proof. In view of Lemma 5.1.7, we try to define recursively a A_∞-functor $\varphi\colon \mathbb{E} \to \underline{\mathrm{Mor}}\,\mathbf{B}$ such that $S\varphi = F, T\varphi = G$, and the induced functor

$$\mathbb{E} = H^0(\mathbb{E}) \to \mathrm{Mor}(H^0(\mathbf{B}))$$

is equal to $\overline{\varphi}$. First, we define a map φ^0 on objects: for any $E \in \mathbb{E}$, we set

$$\varphi^0(E) = (F^0(E), G^0(E), \varphi_E),$$

where φ_E is a chosen lift of the given map $\overline{\varphi}_E \colon F^0(E) \to G^0(E)$. Next, we define φ^1 on a given basis (including the identities of all objects) of the space of morphisms. Given an element $f \colon E_0 \to E_1$ of this basis, we set

$$\varphi^1(f) = (F^1(f), G^1(f), h^1(f)),$$

where $h^1(f)$ is a chosen degree -1 morphism such that

$$-\mu^1_{\mathbf{B}}(h^1(f)) = \mu^2_{\mathbf{B}}(G^1(f), \varphi_{E_0}) - \mu^2_{\mathbf{B}}(\varphi_{E_1}, F^1(f)).$$

$h^1(f)$ exists by the hypothesis that $\overline{\varphi} \colon H^0(F) \to H^0(G)$ is a natural transformation. Moreover, we may choose $h^1(1_E) = 0$ for all $E \in \mathbb{E}$. By construction, $\varphi^1(f)$ is a closed degree 0 morphism in $\mathbf{Q} = \underline{\operatorname{Mor}}\,\mathbf{B}$ (see Example 5.1.4), as required by (5.1.9), and $\varphi^1(1_E) = 1_{\varphi^0(E)}$.

Now, for $d \geq 2$, assume that we have defined a sequence of maps $(\varphi^1, \dots, \varphi^{d-1})$ satisfying (5.1.9) and strict unitality, with

$$\varphi^k(f_k, \dots, f_1) = (F^k(f_k, \dots, f_1), G^k(f_k, \dots, f_1), h^k(f_k, \dots, f_1)).$$

Given maps $f_i \colon E_{i-1} \to E_i$ in our chosen basis for $i = 1, \dots, d$, by Lemma 5.2.1 the expression

$$\begin{aligned} &\sum_{n=0}^{d-2} (-1)^{\maltese_n} \varphi^{d-1}(f_d, \dots f_{n+3}, \mu^2_{\mathbb{E}}(f_{n+2}, f_{n+1}), f_n, \dots, f_1) \\ &\qquad - \sum_{j=1}^{d-1} \mu^2_{\mathbf{Q}}(\varphi^j(f_d, \dots, f_{d-j+1}), \varphi^{d-j}(f_{d-j}, \dots, f_1)) \end{aligned} \tag{5.2.4}$$

is a $\mu^1_{\mathbf{Q}}$-cocycle $(F^0(E_0), G^0(E_0), \varphi_{E_0}) \to (F^0(E_d), G^0(E_d), \varphi_{E_d})$, of degree $1 - (d-1) = 2 - d$. Since F and G are A_∞-functors, we have that

$$(5.2.4) = (\mu^1_{\mathbf{B}}(F^d(f_d, \dots, f_1), \mu^1_{\mathbf{B}}(G^d(f_d, \dots, f_1), \cdots).$$

Then, the condition (5.2.3) allows us to apply Lemma 5.2.2 (with $n = 2 - d$). We may choose $h^d(f_d, \dots f_1)$ such that

$$(5.2.4) = \mu^1_{\mathbf{Q}}(F^d(f_d, \dots, f_1), G^d(f_d, \dots, f_1), h^d(f_d, \dots, f_1)).$$

So, defining

$$\varphi^d(f_d, \dots, f_1) = (F^d(f_d, \dots, f_1), G^d(f_d, \dots, f_1), h^d(f_d, \dots, f_1))$$

we get the correct identity (5.1.9). Notice that, if one of the f_i is an identity morphism, then expression (5.2.4) vanishes, so in that case we may choose $h^d(f_d, \dots, f_1) = 0$, and hence $\varphi^d(f_d, \dots, f_1) = 0$, which is the strict unitality condition. Finally, our result follows by recursion. □

5.3 Applications

In this section we describe an application of the above technique which gives uniqueness results of Fourier-Mukai kernels. The dg-categories of interest in these applications are enhancements of Verdier quotients of the form $\mathsf{D}(\mathbb{A})/L$, where $\mathbb{A}$ is a $\mathbf{k}$-linear category and L is a full subcategory of $\mathsf{D}(\mathbb{A})$ with suitable hypotheses. More precisely, we will work in the framework of the following result, whose proof is essentially contained in [LO10, Section 6, first part].

Lemma 5.3.1. *Let $\mathbb{A}$ be a $\mathbf{k}$-linear category, viewed as a dg-category. Let $L \subseteq \mathsf{D}(\mathbb{A})$ be a localising subcategory (namely, strictly full and closed under direct sums), generated by compact objects $L^c = L \cap \mathsf{D}(\mathbb{A})^c$. There is a canonical functor*

$$\iota\colon \mathbb{A} \hookrightarrow \mathsf{D}(\mathbb{A})^c \to \mathsf{D}(\mathbb{A})^c/L^c \hookrightarrow (\mathsf{D}(\mathbb{A})/L)^c, \tag{5.3.1}$$

where the composition of the last two maps is the restriction of the quotient functor $\mathsf{D}(\mathbb{A}) \to \mathsf{D}(\mathbb{A})/L$. The triangulated category $(\mathsf{D}(\mathbb{A})/L)^c$ is the idempotent completion of $\mathsf{D}(\mathbb{A})^c/L^c$, and it is classically generated by the full subcategory with objects $\iota(\mathbb{A})$. Moreover, if $\mathbf{D}$ together with the equivalence

$$\epsilon\colon (\mathsf{D}(\mathbb{A})/L)^c \to H^0(\mathbf{D})$$

is an enhancement of $(\mathsf{D}(\mathbb{A})/L)^c$, then $\mathbf{D}$ is quasi-equivalent to $\mathrm{per}_{\mathrm{dg}}(\mathbb{A}')$, where $\mathbb{A}'$ is the full dg-subcategory of $\mathbf{D}$ whose object are given by $\epsilon(\iota(\mathbb{A}))$.

Verdier quotients such as $\mathsf{D}(\mathbb{A})^c/L^c$ are enhanced by the *Drinfeld dg-quotient*. We state its definition and main properties, which we will need in the following; they are directly taken from [Dri04, 1.6.2].

Definition 5.3.2. Let $\mathbf{A}$ be a dg-category, and let $\mathbf{B}$ be a full dg-subcategory of $\mathbf{A}$. A *dg-quotient* of $\mathbf{A}$ modulo $\mathbf{B}$ is a dg-category $\mathbf{A}/\mathbf{B}$ together with a quasi-functor $\pi\colon \mathbf{A} \to \mathbf{A}/\mathbf{B}$, such that for any dg-category $\mathbf{C}$ the induced functor

$$\pi^*\colon H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}/\mathbf{B}, \mathbf{C})) \to H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{C})) \tag{5.3.2}$$

is fully faithful, and its essential image consists of quasi-functors $F\colon \mathbf{A} \to \mathbf{C}$ such that $H^0(F)$ maps objects of $\mathbf{B}$ to zero objects in $H^0(\mathbf{C})$.

Theorem 5.3.3. *Let $\mathbf{A}$ be a dg-category, and let $\mathbf{B}$ be a full dg-subcategory of $\mathbf{A}$. Then, a dg-quotient $(\mathbf{A}/\mathbf{B}, \pi)$ exists, and it is uniquely determined up to natural quasi-equivalence. Moreover, if $\mathbf{A}$ is pretriangulated and $H^0(\mathbf{B})$ is a triangulated subcategory of $H^0(\mathbf{A})$, then $(H^0(\mathbf{A}/\mathbf{B}), H^0(\pi))$ is a Verdier quotient of $H^0(\mathbf{A})$ modulo $H^0(\mathbf{B})$:*

$$H^0(\mathbf{A})/H^0(\mathbf{B}) \xrightarrow{\sim} H^0(\mathbf{A}/\mathbf{B}). \tag{5.3.3}$$

Remark 5.3.4. Assume the framework of Lemma 5.3.1. We know that the category $\mathsf{D}(\mathbb{A})^c$ has $\mathrm{per}_{\mathrm{dg}}(\mathbb{A})$ as a dg-enhancement. Moreover, taking $\mathcal{L}^c$ to be the full dg-subcategory of

$\mathrm{per}_{\mathrm{dg}}(\mathbb{A})$ whose objects correspond to L^c, we find out that the dg-quotient $\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c$ is an enhancement of $\mathsf{D}(\mathbb{A})^c/L^c$. Moreover, since $(\mathsf{D}(\mathbb{A})/L)^c$ can be viewed as the idempotent completion of $\mathsf{D}(\mathbb{A})^c/L^c$, we find out that the dg-category

$$\mathrm{per}_{\mathrm{dg}}(\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c)$$

is an enhancement of $(\mathsf{D}(\mathbb{A})/L)^c$. Without loss of generality, we may assume that the above functor ι is obtained in H^0 by the quasi-functor

$$\tilde{\iota}\colon \mathbb{A} \hookrightarrow \mathrm{per}_{\mathrm{dg}}(\mathbb{A}) \xrightarrow{\pi} \mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c \hookrightarrow \mathrm{per}_{\mathrm{dg}}(\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c). \tag{5.3.4}$$

Notice that the quasi-functor $\mathrm{per}_{\mathrm{dg}}(\mathbb{A}) \xrightarrow{\pi} \mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c$ is the canonical projection to the dg-quotient, and the fully faithful dg-functor $\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c \hookrightarrow \mathrm{per}_{\mathrm{dg}}(\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c)$ is the canonical inclusion. They are respectively involved with the universal properties (5.3.2) and (3.5.5).

Now, [LO10, Theorem 2.8] tells us that, under the vanishing hypothesis

$$(\mathsf{D}(\mathbb{A})/L)(\iota(A), \iota(A')[j]) \cong 0, \quad \forall\, j < 0, \quad \forall\, A, A' \in \mathbb{A}, \tag{5.3.5}$$

the category $(\mathsf{D}(\mathbb{A})/L)^c$ admits a *unique* dg-enhancement (up to quasi-equivalence). So, in that case, we are allowed to identify any such enhancement $\mathbf{D}$, up to quasi-equivalence, to the dg-category $\mathrm{per}_{\mathrm{dg}}(\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c)$.

Now, the abstract result of the previous section allows us to prove the following:

Theorem 5.3.5. *Assume the framework of Lemma 5.3.1, and assume that* $(\mathsf{D}(\mathbb{A})/L)^c$ *has a unique enhancement. Let* $\mathbf{D}$ *be such an enhancement, and for simplicity identify* $H^0(\mathbf{D}) = (\mathsf{D}(\mathbb{A})/L)^c$. *Let* $F, G\colon \mathbf{D} \to \mathbf{B}$ *be quasi-functors taking values in a triangulated dg-category* $\mathbf{B}$*, both satisfying the vanishing hypothesis:*

$$H^0(\mathbf{B})(F(\iota(A)), F(\iota(A'))[j]) \cong 0, \quad \forall\, j < 0, \tag{5.3.6}$$

for all $A, A' \in \mathbb{A}$. *Then, if*

$$H^0(F) \circ \iota \cong H^0(G) \circ \iota\colon \mathbb{A} \to \mathbf{B},$$

we have that $F \cong G$ *as quasi-functors.*

Proof. Recalling Remark 5.3.4, we are allowed to identify $\mathbf{D}$ to $\mathrm{per}_{\mathrm{dg}}(\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c)$. By the universal property of $\mathrm{per}_{\mathrm{dg}}(\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c)$, we have that $F \cong G$ if and only if $F_{|\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c} \cong G_{|\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c}$. Then, by the universal property of the dg-quotient, this is equivalent to

$$F_{|\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c} \circ \pi \cong G_{|\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c} \circ \pi\colon \mathrm{per}_{\mathrm{dg}}(\mathbb{A}) \to \mathbf{B}.$$

Finally, by the universal property of $\mathrm{per}_{\mathrm{dg}}(\mathbb{A})$, this is equivalent to

$$F \circ \tilde{\iota} \cong G \circ \tilde{\iota}\colon \mathbb{A} \to \mathbf{B}.$$

Now, recalling that we have identified $\iota = H^0(\tilde{\iota})$, a direct application of Theorem 5.2.4 gives the desired result. □

The above result has an interesting application. Let X be a quasi-projective scheme, viewed as open subscheme of a projective scheme $\overline{X}$. Then, the derived category $\mathfrak{D}(\mathrm{QCoh}(X))$ of quasi-coherent sheaves on X can be described as a quotient $\mathsf{D}(\mathbb{A})/L$. Namely, take $\mathbb{A}$ as the category with objects given by the integers, and

$$\mathbb{A}(i,j) = H^0(\overline{X}, \mathcal{O}_{\overline{X}}(j-i)), \tag{5.3.7}$$

with composition induced by that of the graded algebra $\bigoplus_n H^0(\overline{X}, \mathcal{O}_{\overline{X}}(n))$. The subcategory L is taken to be the category of all objects in $\mathsf{D}(\mathbb{A})$ whose cohomologies are "I-torsion modules" (for details, see [LO10, before Corollary 7.8]). It can be proved that there is an equivalence $\mathsf{D}(\mathbb{A})/L \cong \mathfrak{D}(\mathrm{QCoh}(X))$, and also that the natural functor

$$\mathbb{A} \hookrightarrow \mathsf{D}(\mathbb{A}) \to \mathsf{D}(\mathbb{A})/L \xrightarrow{\sim} \mathfrak{D}(\mathrm{QCoh}(X))$$

maps any integer $j \in \mathrm{Ob}\,\mathbb{A}$ to the sheaf $\mathcal{O}_X(j)$. Moreover, the subcategory L satisfies the hypotheses of Lemma 5.3.1, and in particular the above discussion restricts to compact objects and perfect complexes. Namely, we have an equivalence $(\mathsf{D}(\mathbb{A})/L)^c \cong \mathrm{Perf}(X)$ which, composed with the functor (5.3.1), gives:

$$\begin{aligned} &\mathbb{A} \xrightarrow{\iota} (\mathsf{D}(\mathbb{A})/L)^c \xrightarrow{\sim} \mathrm{Perf}(X), \\ &j \mapsto \mathcal{O}_X(j). \end{aligned} \tag{5.3.8}$$

Now, let $\mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(X))$ be an enhancement of $\mathfrak{D}(\mathrm{QCoh}(X))$, and for simplicity identify this category to $H^0(\mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(X)))$. Recall that the full dg-subcategory $\mathrm{Perf}_{\mathrm{dg}}(X)$ of $\mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(X))$ whose objects are the compact objects in $\mathfrak{D}(\mathrm{QCoh}(X))$ is an enhancement of $\mathrm{Perf}(X)$; also, recall that these enhancements are uniquely determined, by [LO10, Corollary 7.8, Theorem 7.9]. Upon identifying $(\mathsf{D}(\mathbb{A})/L)^c$ to $\mathrm{Perf}(X)$ via the equivalence discussed above, we immediately get the following:

Corollary 5.3.6. *Let X be a quasi-projective scheme, and let $\mathbf{B}$ be a triangulated dg-category. Let $F, G\colon \mathrm{Perf}_{\mathrm{dg}}(X) \to \mathbf{B}$ be quasi-functors which satisfy the vanishing condition*

$$H^0(\mathbf{B})(F(\mathcal{O}_X(n)), F(\mathcal{O}_X(m))[j]) = 0, \quad \forall\, j < 0,$$

for all $n, m \in \mathbb{Z}$. Then, if $H^0(F) \cong H^0(G)$, we have that $F \cong G$ as quasi-functors.

Finally, we apply this machinery to the uniqueness problem of Fourier-Mukai kernels, as explained in Section 0.2, hence obtaining the following uniqueness result:

Theorem 5.3.7. *Let X and Y be schemes satisfying the hypotheses of both Theorems 0.2.2 and 0.2.1, with X quasi-projective. Let $\mathcal{E}, \mathcal{E}' \in \mathfrak{D}(\mathrm{QCoh}(X \times Y))$ be such that*

$$\Phi_{\mathcal{E}}^{X \to Y} \cong \Phi_{\mathcal{E}'}^{X \to Y} \cong F\colon\ \mathrm{Perf}(X) \to \mathfrak{D}(\mathrm{QCoh}(Y)),$$

and $\mathrm{Hom}(F(\mathcal{O}_X(n)), F(\mathcal{O}_X(m))[j]) = 0$ for all $j < 0$, for all $n, m \in \mathbb{Z}$. Then $\mathcal{E} \cong \mathcal{E}'$.

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Contents

List of notations

$\mathrm{rep}^r(\mathbf{A},\mathbf{B})$ The dg-category of right representable **A**-**B**-dg-bimodules, page 56

$1_{(A,n,0)}, 1_{(A,0,n)}$ The morphisms associated to the n-shift of an object A in a dg-category, page 34

$\mathrm{tria}(\mathbf{A})$ The triangulated category associated to a dg-category **A**, page 48

$A \approx A'$ Objects A, A' in a dg-category are homotopy equivalent, page 8

$A \cong A'$ Objects A, A' in a dg-category are dg-isomorphic, page 8

$A[n]$ The n-shift of an object A in a dg-category, page 34

$F\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ Notation which means that F is a **A**-**B**-dg-bimodule, or $F \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A},\mathbf{B})$, page 57

$F \overset{\mathrm{qis}}{\approx} G$ Dg-modules F and G are quasi-isomorphic, page 42

$F^B_{A_1,A_2}$ Einstein notation for $F(B, A_1, A_2)$, meaning that the functor F is covariant in A_1, A_2 and contravariant in B, page 6

$G \diamond F$ Composition of dg-bimodules, page 57

$H^*(\mathbf{A})$ The graded homotopy category of a dg-category **A**, page 7

$H^0(\mathbf{A})$ The homotopy category of a dg-category **A**, page 7

h^F, h_G, h^F_G Dg-modules induced by dg-functors, page 11

$h_{\mathbf{A}} = h$ The diagonal bimodule of a dg-category **A**, page 11

$K^* = \mathrm{Res}_K$ The restriction functor along K, page 32

$P(\mathbf{A})$ The path object of **A** in the model category of dg-categories, page 18

$V \otimes A$ The tensor or copower of a complex V and an object A of a dg-category, page 30

$Z^*(\mathbf{A})$ The graded underlying category of a dg-category **A**, page 7

$Z^0(\mathbf{A})$ The underlying category of a dg-category **A**, page 7