Perfect complexes of twisted sheaves and dg-enhancements

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Twisted sheaves ...

- In algebraic geometry one encounters problems which can be solved *locally*, but those local solutions *don't glue nicely*. One can try to understand the (co-homological) obstruction to that failure of gluing. This leads to the notion of *twisted sheaf*. A reference here is [1].
- Let X be a scheme, and let $\alpha \in H^2_{\text{ét}}(X, \mathcal{O}_X^*)$. We choose an étale covering $\mathfrak{U} = \{U_i \to X\}$ such that α can be represented by a Čech 2-cocycle $(a_{ijk})_{i,j,k}$ on $\mathfrak{U}$. An *α-twisted sheaf* $\mathcal{F} = (\mathcal{F}_i, \varphi_{ij})$ is then given by the following data:
  - For all $i \in I$, a sheaf $\mathcal{F}_i$ of $\mathcal{O}_{U_i}$-modules.
  - Isomorphisms $\varphi_{ij} \colon \mathcal{F}_j|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_i|_{U_{ij}}$ satisfying twisted cocycle conditions:

  $$\varphi_{ii} = 1,\ \varphi_{ji} = \varphi_{ij}^{-1},\ \varphi_{ik}|_{U_{ijk}} = a_{ijk}(\varphi_{ij} \circ \varphi_{jk})|_{U_{ijk}}.$$

  The definition can be seen to be independent of the choice of the representative of α and the covering $\mathfrak{U}$.
- More generally, let X be a DM-stack, and let $\alpha \in H^2(X_{\text{ét}}, \mathcal{O}_X^*)$. Let $U_0 \to X$ be an étale atlas, and set: $U_1 = U_0 \times_X U_0$, $U_2 = U_0 \times_X U_0 \times_X U_0$, with natural projections $p_0, p_1 \colon U_1 \to U_0$ and $p_{01}, p_{12}, p_{02} \colon U_2 \to U_1$. We choose a cocycle $a \in \Gamma(U_2, \mathcal{O}_{U_2}^*)$ which represents α. An *α-twisted sheaf* $\mathcal{F}$ is then given by the following data:
  - A sheaf $\mathcal{F}_0$ of $\mathcal{O}_{U_0}$-modules.
  - A gluing datum $\varphi \colon p_0^*\mathcal{F}_0 \xrightarrow{\sim} p_1^*\mathcal{F}_0$ satisying a twisted cocycle condition:

  $$p_{02}^*\varphi = a(p_{12}^*\varphi \circ p_{01}^*\varphi).$$

... and their (derived) categories

- We denote by $\operatorname{Mod}(X, \alpha)$ the abelian category of α-twisted sheaves. Every definition for ordinary sheaves which is "local" can be extended to α-twisted sheaves. For example, a sheaf $\mathcal{F} \in \operatorname{Mod}(X, \alpha)$ is *quasi-coherent* if the underlying sheaf $\mathcal{F}_0$ of $\mathcal{O}_{U_0}$-modules is quasi-coherent. Given $\mathcal{F} \in \operatorname{Mod}(X, \alpha)$ and $\mathcal{G} \in \operatorname{Mod}(X, \beta)$, we may define in this manner the tensor product and the sheaf hom:

  $$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \in \operatorname{Mod}(X, \alpha\beta), \quad \mathcal{H}om(\mathcal{F}, \mathcal{G}) \in \operatorname{Mod}(X, \alpha^{-1}\beta),$$

  and also "twisted" pushforward and pullback functors.
- We denote by $\mathfrak{D}(X, \alpha)$ the derived category of $\operatorname{Mod}(X, \alpha)$, namely the localization of the homotopy category of complexes $\mathfrak{K}(X, \alpha)$ along quasi-isomorphisms. We also denote by $\mathfrak{D}_{\mathrm{qc}}(X, \alpha)$ its full subcategory of complexes with quasi-coherent cohomology.
- A complex $\mathcal{F}^\bullet \in \mathfrak{D}(X, \alpha)$ is *K-injective* if $\mathfrak{K}(X, \alpha)(\mathcal{A}^\bullet, \mathcal{F}^\bullet) = 0$ for any acyclic complex $\mathcal{A}^\bullet \in \mathfrak{D}(X, \alpha)$. It is *K-flat* if $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{A}^\bullet$ is acyclic for any acyclic complex $\mathcal{A}^\bullet \in \mathfrak{D}(X, \beta)$, for any $\beta \in H^2(X_{\text{ét}}, \mathcal{O}_X^*)$.

  Lemma. *Every complex $\mathcal{F}^\bullet$ has a K-injective resolution $\mathcal{F}^\bullet \to R(\mathcal{F}^\bullet)$ and a K-flat resolution $Q(\mathcal{F}^\bullet) \to \mathcal{F}^\bullet$.*

  Thanks to K-injective and K-flat resolutions, we can define derived functors just as in the untwisted setting, for instance:

  $$\mathcal{F}^\bullet \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet = Q(\mathcal{F}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet,$$
  $$\mathbb{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathcal{H}om(\mathcal{F}^\bullet, R(\mathcal{G}^\bullet)).$$

Perfect complexes and compact objects

- Following [3], a complex $\mathcal{F}^\bullet \in \mathfrak{D}_{\mathrm{qc}}(X)$ is *perfect* if for any commutative ring R and any smooth morphism $f \colon \operatorname{Spec}(R) \to X$, the complex of R-modules $\mathbb{R}\Gamma(\operatorname{Spec}(R), \mathbb{L}f^*(\mathcal{F}^\bullet))$ is perfect. We denote by $\operatorname{Perf}(X) \subseteq \mathfrak{D}_{\mathrm{qc}}(X)$ the full subcategory of perfect complexes.
- A complex of α-twisted sheaves $\mathcal{F}^\bullet \in \mathfrak{D}_{\mathrm{qc}}(X, \alpha)$ is *perfect* if, given an étale atlas $p \colon U_0 \to X$ where α is represented, we have $\mathbb{L}p^*(\mathcal{F}^\bullet) \in \operatorname{Perf}(U_0)$.
- A complex $\mathcal{F}^\bullet \in \mathfrak{D}_{\mathrm{qc}}(X, \alpha)$ is *compact* if for any family of objects $(\mathcal{E}_k^\bullet)_k$ in $\mathfrak{D}_{\mathrm{qc}}(X, \alpha)$, the natural morphism $\bigoplus_k \operatorname{Hom}(\mathcal{F}^\bullet, \mathcal{E}_k^\bullet) \to \operatorname{Hom}(\mathcal{F}^\bullet, \bigoplus_k \mathcal{E}_k^\bullet)$ is an isomorphism.
- It is known [3] that if X is *concentrated*, i.e. quasi-compact, quasi-separated and such that $\mathcal{O}_X$ is a compact object, then perfect complexes and compact objects coincide: $\operatorname{Perf}(X) = \mathfrak{D}_{\mathrm{qc}}(X)^c$.

Theorem. *Let X be a concentrated DM-stack, and let $\alpha \in H^2(X_{\text{ét}}, \mathcal{O}_X^*)$. Then, the perfect complexes of α-twisted sheaves are the compact objects of $\mathfrak{D}_{\mathrm{qc}}(X, \alpha)$:*

$$\operatorname{Perf}(X, \alpha) = \mathfrak{D}_{\mathrm{qc}}(X, \alpha)^c.$$

Idea of proof. Assuming that $\mathcal{P}^\bullet$ is perfect, we may prove that it is *dualizable*, namely there is a natural isomorphism $\mathbb{R}\mathcal{H}om(\mathcal{P}^\bullet, \mathcal{O}_X) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{F}^\bullet \cong \mathbb{R}\mathcal{H}om(\mathcal{P}^\bullet, \mathcal{F}^\bullet)$. From this, using that $\mathcal{O}_X$ is compact, we conclude that $\mathcal{P}^\bullet$ is compact.

On the other hand, if $\mathcal{P}^\bullet$ is compact we can show that it is *locally compact*, namely it is compact if restricted to the chosen étale atlas $U_0 \to X$. We are now in the untwisted setting, and we may conclude that $\mathcal{P}^\bullet$ is perfect. □

Dg-enhancements

- A *dg-enhancement* of a triangulated category $\mathcal{T}$ is a pretriangulated dg-category $\mathcal{A}$ such that $H^0(\mathcal{A}) \cong \mathcal{T}$. The typical question one asks is: does a given $\mathcal{T}$ have a dg-enhancement? Is this enhancement *unique*, in the sense that given dg-enhancements $\mathcal{A}$ and $\mathcal{A}'$, they are quasi-equivalent as dg-categories?
- If $\mathcal{G}$ is a Grothendieck abelian category, then $\mathfrak{D}(\mathcal{G})$ has a unique enhancement [2, Theorem A].
- Furthermore [2, Theorem B], assume that $\mathcal{G}$ has a small set $\mathcal{A}$ of generators such that:
  - $\mathcal{A}$ is closed under finite coproducts.
  - Every object of $\mathcal{A}$ is a Noetherian object in $\mathcal{G}$.
  - If $f \colon A \to A'$ is an epimorphism of $\mathcal{G}$ with $A, A' \in \mathcal{A}$, then $\ker f \in \mathcal{A}$.
  - For every $A \in \mathcal{A}$ there exists $N(A) > 0$ such that $\mathfrak{D}(\mathcal{G})(A, A'[N(A)]) = 0$ for every $A' \in \mathcal{A}$.

  Then, the subcategory of compact objects $\mathfrak{D}(\mathcal{G})^c$ has a unique enhancement.
- In particular [2, Proposition 6.10], if X is a Noetherian concentrated algebraic stack with quasi-finite affine diagonal and such that $\operatorname{QCoh}(X)$ is generated by a set contained in $\operatorname{Coh}(X) \cap \operatorname{Perf}(X)$, then $\operatorname{Perf}(X)$ has a unique dg-enhancement.

Theorem (Under construction!). *Let X be a Noetherian concentrated DM-stack with quasi-finite affine diagonal and such that $\operatorname{QCoh}(X)$ is generated by a set contained in $\operatorname{Coh}(X) \cap \operatorname{Perf}(X)$. Moroever, let $\alpha \in H^2(X_{\text{ét}}, \mathcal{O}_X^*)$ and assume that there exists a locally free α-twisted sheaf of finite rank. Then, $\operatorname{Perf}(X, \alpha)$ has a unique dg-enhancement.*

Idea of proof. From the assumptions on X, we deduce that $\mathfrak{D}_{\mathrm{qc}}(X, \alpha) = \mathfrak{D}(\operatorname{QCoh}(X, \alpha))$. We can now apply [2, Theorem B], recalling that perfect complexes of twisted sheaves coincide with the compact objects. □

Essential bibliography

[1] Andrei Horia Caldararu, *Derived categories of twisted sheaves on Calabi-Yau manifolds*, ProQuest LLC, Ann Arbor, MI, 2000, Thesis (Ph.D.)–Cornell University.

[2] Alberto Canonaco and Paolo Stellari, *Uniqueness of dg enhancements for the derived category of a Grothendieck category*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 11, 2607–2641.

[3] Jack Hall and David Rydh, *Perfect complexes on algebraic stacks*, Compos. Math. **153** (2017), no. 11, 2318–2367.