A DERIVED GABRIEL-POPESCU THEOREM FOR T-STRUCTURES VIA DERIVED INJECTIVES

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T-STRUCTURES

We fix an *enhanced triangulated category* $\mathcal{T}$. More precisely, we think of $\mathcal{T}$ as $H^0(\mathcal{A})$ for a fixed pretriangulated dg-category $\mathcal{A}$, which is called a *dg-enhancement* – other choices of enhancements are possible (stable ∞-categories, stable derivators).

A *t-structure*, in some sense, is a "triangulated version" of a torsion theory:

Definition. A *t-structure* on $\mathcal{T}$ is a pair of full subcategories $(\mathcal{T}_{\leq 0}, \mathcal{T}_{\geq 1})$ such that:

- $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}_{\leq 0}, \mathcal{T}_{\geq 1}) = 0$.
- $\mathcal{T}_{\leq 0}$ and $\mathcal{T}_{\geq 1}$ are closed under extensions and direct summands.
- $\mathcal{T}_{\leq 0}[1] \subseteq \mathcal{T}_{\leq 0}$ and $\mathcal{T}_{\geq 1}[-1] \subseteq \mathcal{T}_{\geq 1}$.
- For all $A \in \mathcal{T}$ and for all $n \in \mathbb{Z}$, there is a (functorial) distinguished triangle
$$\tau_{\leq n} A \to A \to \tau_{\geq n+1} A \to (\tau_{\leq n} A)[1],$$
with $\tau_{\leq n} A \in \mathcal{T}_{\leq n} = \mathcal{T}_{\leq 0}[-n]$ and $\tau_{\geq n+1} A \in \mathcal{T}_{\geq n+1} = \mathcal{T}_{\geq 1}[-n]$.

- $\tau_{\leq n}(-)$ and $\tau_{\geq n+1}(-)$ are called *(left or right) truncations*. They yield functors:
$$\tau_{\leq n} \colon \mathcal{T} \to \mathcal{T}_{\leq n}, \qquad \tau_{\geq n+1} \colon \mathcal{T} \to \mathcal{T}_{\geq n+1}.$$
- We define
$$\mathcal{T}^{\heartsuit} = \mathcal{T}_{\leq 0} \cap \mathcal{T}_{\geq 0}.$$
This is called the *heart* of the t-structure, and it is an abelian category.
- We define
$$H^0_t = \tau_{\leq 0} \tau_{\geq 0} \colon \mathcal{T} \to \mathcal{T}^{\heartsuit}, \qquad H^n_t(-) = H^0(-[n]).$$
The functor H^0_t is cohomological.

We say that the given t-structure is *non-degenerate* if $A \cong 0$ is equivalent to $H^n_t(A) \cong 0$ for all $n \in \mathbb{Z}$.

DERIVED INJECTIVE OBJECTS

We fix an enhanced triangulated category $\mathcal{T}$ with a t-structure $(\mathcal{T}_{\leq 0}, \mathcal{T}_{\geq 1})$.

Principle. Enhanced triangulated categories *with t-structures* allow us to generalize many results and constructions of abelian categories to the derived framework.

Definition. An object $I \in \mathcal{T}$ is *derived injective* if $I \in \mathcal{T}_{\geq 0}$, and for any $Z \in \mathcal{T}_{\geq 0}$ we have:
$$\mathrm{Ext}^1_{\mathcal{T}}(Z, I) = \mathrm{Hom}_{\mathcal{T}}(Z[-1], I) = 0.$$

- If I is derived injective, then $H^0_t(I)$ is an injective object in $\mathcal{T}^{\heartsuit}$.
- For all $X \in \mathcal{T}$, the H^0_t functor induces an isomorphism:
$$\mathcal{T}(X, I) \xrightarrow{\sim} \mathcal{T}^{\heartsuit}(H^0_t(X), H^0_t(I)).$$
- We say that $\mathcal{T}$ *has enough derived injectives* if for any object $A \in \mathcal{T}_{\geq 0}$ there is a morphism $A \to I$ with I derived injective, such that the induced morphism $H^0_t(A) \to H^0_t(I)$ is a monomorphism in $\mathcal{T}^{\heartsuit}$.
- If $\mathcal{T}$ has nice properties (in particular, it has enough derived injectives and it is non-degenerate), its "left bounded part" $\mathcal{T}^+$ can be completely recovered by suitable *derived injective resolutions*.

  An example of this is the classical equivalence
$$\mathsf{K}^+(\mathrm{Inj}(\mathfrak{A})) \cong \mathsf{D}^+(\mathfrak{A}),$$
if $\mathfrak{A}$ is an abelian category with enough injectives.

THE DERIVED GABRIEL-POPESCU THEOREM

The (classical) Gabriel-Popescu theorem exhibits any Grothendieck abelian category as a localization of the category of modules over a suitable ring. We generalize this to a *derived Gabriel-Popescu theorem*, which exhibits any enhanced triangulated category with a "Grothendieck-like" t-structure as a localization (in the sense of t-structures) of the derived category of a suitable dg-algebra.

Setup. We let $\mathcal{T}$ be an enhanced triangulated category endowed with a non-degenerate t-structure with suitable properties making it "Grothendieck-like". In particular:

- $\mathcal{T}$ is cocomplete, namely, it has arbitrary (small) direct sums. Moreover, the cohomological functor $H^0_t(-)$ preserves direct sums.
- $\mathcal{T}$ has a generator $U \in \mathcal{T}_{\leq 0}$, such that for any object $A \in \mathcal{T}_{\leq 0}$ there is a morphism $U^{\oplus I} \to A$ such that the induced morphism $H^0_t(U)^{\oplus I} \twoheadrightarrow H^0_t(A)$ is an epimorphism in $\mathcal{T}^{\heartsuit}$.
- We set $R = \tau_{\leq 0} \operatorname{RHom}_{\mathcal{T}}(U, U)$. By $\operatorname{RHom}_{\mathcal{T}}(U, U)$ we denote the dg-algebra of endomorphisms $U \to U$ in our choice of dg-enhancement of $\mathcal{T}$, and $\tau_{\leq 0}$ is the smart truncation.
- The derived category $\mathsf{D}(R)$ has a natural non-degenerate t-structure with heart $\mathrm{Mod}(H^0(R))$. The truncation functors are the usual "smart" truncations of dg-modules. It is important here that R is concentrated in nonpositive degrees.
- $\mathcal{T}$ has enough derived injectives and the heart $\mathcal{T}^{\heartsuit}$ has exact filtered colimits. In particular, $\mathcal{T}^{\heartsuit}$ is a Grothendieck abelian category.

Theorem ([1]). *Let $\mathcal{T}$ be an enhanced triangulated category endowed with a t-structure as in the above setup. The functor*
$$G \colon \mathcal{T} \to \mathsf{D}(R), \qquad A \mapsto \operatorname{RHom}_{\mathcal{T}}(U, A).$$
is fully faithful and has a t-exact left adjoint $F \colon \mathsf{D}(R) \to \mathfrak{A}$. "t-exact" means that $F(\mathsf{D}(R)_{\leq 0}) \subseteq \mathcal{T}_{\leq 0}$ and $F(\mathsf{D}(R)_{\geq 1}) \subseteq \mathcal{T}_{\geq 1}$

Idea of proof. We generalize the "quick proof" by Mitchell [2] of the classical Gabriel-Popescu theorem. In particular, we can prove t-exactness of F by checking that G preserves derived injective objects. $\square$

ESSENTIAL BIBLIOGRAPHY

[1] Francesco Genovese and Julia Ramos González, *A Derived Gabriel–Popescu Theorem for t-Structures via Derived Injectives*, International Mathematics Research Notices (2022), rnab367.

[2] Barry Mitchell, *A quick proof of the Gabriel-Popesco theorem*, J. Pure Appl. Algebra **20** (1981), no. 3, 313–315.