THE UNIQUENESS PROBLEM OF DG-LIFTS AND FOURIER-MUKAI KERNELS

FRANCESCO GENOVESE

Abstract. We address the uniqueness problem of dg-lifts of exact functors between triangulated categories, and its relationship with the uniqueness problem of Fourier-Mukai kernels. We prove a positive result under a vanishing hypothesis on the functors, using A_∞-categorical techniques.

1. Introduction

Triangulated categories are nowadays a classical topic in mathematics, with many applications in geometry and algebra. In particular, they arise in algebraic geometry as derived categories of (quasi-)coherent sheaves on schemes. Their serious technical drawbacks (in particular, the non-functoriality of cones) suggest that they are actually "shadows" of more complicated, higher categorical structures. A popular way to enhance the understanding of triangulated categories is to use *differential graded (dg-) categories*, namely categories enriched in complexes of modules over a ground field $\mathbf{k}$. A *(dg-)enhancement* of a triangulated category $\mathbf{T}$ is a pretriangulated dg-category $\mathbf{A}$ such that $H^0(\mathbf{A})$ is equivalent to $\mathbf{T}$; with the term *pretriangulated dg-category* we mean a dg-category which, roughly speaking, contains shifts and functorial cones up to homotopy equivalence. If $\mathbf{A}$ is a pretriangulated dg-category, then its *homotopy category* $H^0(\mathbf{A})$ – obtained by taking zeroth cohomology – has a natural structure of triangulated category; a dg-functor $F\colon \mathbf{A} \to \mathbf{B}$ between pretriangulated dg-categories (which is simply a functor of enriched categories) induces an exact functor $H^0(F)\colon H^0(\mathbf{A}) \to H^0(\mathbf{B})$. Unfortunately, the notion of dg-functor is too "strict" and doesn't behave well within the homotopy theory of dg-categories; so, we must consider more complicated – homotopy relevant – replacements, namely *quasi-functors*. They can be described concretely as *right quasi-representable bimodules* (see Proposition 2.9) or as A_∞*-functors* (see Proposition 4.9).

Quasi-functors $\mathbf{A} \to \mathbf{B}$ form a dg-category, which is denoted by $\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B})$. This is the internal hom in the homotopy category of dg-categories Hqe, which is obtained localising the model category dgCat of (small) dg-categories along quasi-equivalences (see Theorem 2.8). Like dg-functors, quasi-functors yield ordinary functors between the homotopy categories, namely, there is a functor:

$$H^0 = \Phi^{\mathbf{A}\to\mathbf{B}}\colon H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B})) \to \mathrm{Fun}(H^0(\mathbf{A}), H^0(\mathbf{B})). \tag{1.1}$$

If $\mathbf{A}$ and $\mathbf{B}$ are pretriangulated, then $\Phi^{\mathbf{A}\to\mathbf{B}}$ takes values in the category of exact functors $\mathrm{Fun}_{\mathrm{ex}}(H^0(\mathbf{A}), H^0(\mathbf{B}))$. By definition, a *dg-lift* of an exact functor $\overline{F}\colon H^0(\mathbf{A}) \to H^0(\mathbf{B})$ is a quasi-functor $F\colon \mathbf{A} \to \mathbf{B}$ such that $H^0(F) \cong \overline{F}$. The *uniqueness problem of dg-lifts*, which is the main topic of this work, asks whether two quasi-functors $F, G\colon \mathbf{A} \to \mathbf{B}$ are isomorphic if there is an isomorphism $H^0(F) \cong H^0(G)$. The relevance of this problem lies in the fact that, in the geometric cases, it is essentially equivalent to the *uniqueness problem of Fourier-Mukai kernels*. Let us make this claim precise.

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Let X be a quasi-compact and quasi-separated scheme (over $\mathbf{k}$). We denote by $\mathfrak{D}(\mathrm{QCoh}(X))$ the derived category of quasi-coherent sheaves on X. The subcategory of compact objects of $\mathfrak{D}(\mathrm{QCoh}(X))$ coincides with the category of perfect complexes $\mathrm{Perf}(X)$. Given two schemes X and Y, there is a functor:

$$\Phi^{X\to Y}_{-} \colon \mathfrak{D}(\mathrm{QCoh}(X\times Y)) \to \mathrm{Fun}_{\mathrm{ex}}(\mathrm{Perf}(X), \mathfrak{D}(\mathrm{QCoh}(Y))), \tag{1.2}$$

which maps a complex $\mathcal{E} \in \mathfrak{D}(\mathrm{QCoh}(X\times Y))$ to its *Fourier-Mukai functor*

$$\Phi^{X\to Y}_{\mathcal{E}} = \Phi_{\mathcal{E}} \colon \mathrm{Perf}(X) \to \mathfrak{D}(\mathrm{QCoh}(Y)),$$

which is defined by:

$$\Phi_{\mathcal{E}}(-) = \mathbb{R}(p_2)_*(\mathcal{E} \otimes^{\mathbb{L}} p_1^*(-)),$$

where $p_1 \colon X\times Y \to X$ and $p_2 \colon X\times Y \to Y$ are the natural projections. If an exact functor $F\colon \mathrm{Perf}(X) \to \mathfrak{D}(\mathrm{QCoh}(Y))$ is such that $F \cong \Phi_{\mathcal{E}}$, we say thay $\mathcal{E}$ is a *Fourier-Mukai kernel* of F. Current research is devoted to investigating the properties of $\Phi^{X\to Y}_{-}$ (see [2] for a survey); for instance, the uniqueness problem of Fourier-Mukai kernels amounts to studying if $\Phi^{X\to Y}_{\mathcal{E}} \cong \Phi^{X\to Y}_{\mathcal{E}'}$ implies $\mathcal{E} \cong \mathcal{E}'$. In general, we know that this is false: we can find counterexamples when $X = Y$ is an elliptic curve (see [3]). However, we do obtain a positive answer in some particular cases: for instance, it is known that fully faithful functors $F\colon \mathrm{Perf}(X) \to \mathfrak{D}(\mathrm{QCoh}\, Y)$ admit uniquely determined Fourier-Mukai kernels if X and Y are smooth projective (see [13, Theorem 2.2] for the original formulation).

Now, let us see how this is related to dg-categories and quasi-functors. If X is a scheme (over $\mathbf{k}$), then the derived category $\mathfrak{D}(\mathrm{QCoh}(X))$ has an enhancement, which we call $\mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(X))$, choosing it once and for all and identifying $H^0(\mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(X))) = \mathfrak{D}(\mathrm{QCoh}(X))$. Taking the dg-subcategory of $\mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(X))$ whose objects correspond to $\mathrm{Perf}(X)$, we find an enhancement $\mathrm{Perf}_{\mathrm{dg}}(X)$ of the category of perfect complexes. A remarkable theorem by B. Toën tells us that, under suitable hypotheses, *every quasi-functor has a unique Fourier-Mukai kernel*, in the following sense:

Theorem 1.1 ((Adapted from [17, Theorem 8.9 and Theorem 7.2.1]))**.** *Let X and Y be quasi-compact and separated schemes over $\mathbf{k}$. Then, there is an isomorphism in* Hqe*:*

$$\mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(X\times Y)) \xrightarrow{\sim} \mathbb{R}\underline{\mathrm{Hom}}(\mathrm{Perf}_{\mathrm{dg}}(X), \mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(Y)). \tag{1.3}$$

Combining this with [11, Theorem 1.1], which clarifies a claim in [17, after Corollary 8.12], we get the desired "bridge" between Fourier-Mukai functors and quasi-functors between dg-categories:

Theorem 1.2. *Let X and Y be Noetherian separated schemes over $\mathbf{k}$ such that $X\times Y$ is Noetherian and the following condition holds for both X and Y: any perfect complex is isomorphic to a strictly perfect complex (i. e. a bounded complex of vector bundles). Then, there is a commutative diagram (up to isomorphism):*

$$\begin{array}{ccc} \mathfrak{D}(\mathrm{QCoh}(X\times Y)) & \xrightarrow{\sim} & H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathrm{Perf}_{\mathrm{dg}}(X), \mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(Y))) \\ & \searrow^{\Phi^{X\to Y}_{-}} & \downarrow{\scriptstyle \Phi^{\mathrm{Perf}_{\mathrm{dg}}(X)\to \mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(Y))}} \\ & & \mathrm{Fun}_{\mathrm{ex}}(\mathrm{Perf}(X), \mathfrak{D}(\mathrm{QCoh}\, Y)), \end{array} \tag{1.4}$$

where the horizontal equivalence is induced by (1.3)*.*

Remark 1.3. The hypotheses of the above theorem are satisfied if both X and Y are quasi-projective.

The above result tells us that, under suitable hypotheses, the properties of $\Phi_{-}^{X\to Y}$ are directly translated to those of $\Phi^{\mathrm{Perf}_{\mathrm{dg}}(X)\to\mathfrak{D}_{\mathrm{dg}}(\mathrm{QCoh}(Y))}$. In particular, the uniqueness problem of dg-lifts for functors $\mathrm{Perf}(X)\to\mathfrak{D}(\mathrm{QCoh}(Y))$ (with the above chosen dg-enhancements) is equivalent to the uniqueness problem of Fourier-Mukai kernels.

Next, we state the main theorem of the work, which is purely algebraic. Its proof uses the description of quasi-functors by means of A_∞-functors; even if it involves some rather intricate computations with the A_∞-formalism, it is conceptually not difficult.

Theorem 1.4 ((Theorem 4.15)). *Let $\mathbf{A}$ and $\mathbf{B}$ be triangulated dg-categories, namely pretriangulated dg-categories such that their homotopy categories are idempotent complete. Assume that $\mathbf{A}$ is the* triangulated hull *(see Subsection 2.4) of a $\mathbf{k}$-linear category $\mathbb{E}$. Moreover, let $F,G\colon \mathbf{A}\to\mathbf{B}$ be quasi-functors such that F satisfies the following vanishing conditions:*

$$H^j(\mathbf{B}(F(E),F(E'))) = 0,$$

for all $j<0$ and all $E,E'\in\mathbb{E}$. Then $H^0(F_{|\mathbb{E}})\cong H^0(G_{|\mathbb{E}})$ implies $F\cong G$.

From this theorem, we obtain a result giving uniqueness of dg-lifts in the case where the source dg-category is an enhancement of the subcategory of compact objects in a suitable Verdier quotient of the derived category of a $\mathbf{k}$-linear category (see Theorem 5.5). This applies in particular to $\mathrm{Perf}(X)$, when X is a quasi-projective scheme; from this we obtain the following geometric application:

Theorem 1.5 ((Theorem 5.7)). *Let X and Y be schemes satisfying the hypotheses of Theorem 1.2, with $X\subseteq\mathbb{P}^N$ quasi-projective; for all $n\in\mathbb{Z}$ we denote by $\mathcal{O}_X(n)$ the restriction on X of the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^N$. Let $\mathcal{E},\mathcal{E}'\in\mathfrak{D}(\mathrm{QCoh}(X\times Y))$ be such that*

$$F := \Phi_{\mathcal{E}}^{X\to Y}\cong\Phi_{\mathcal{E}'}^{X\to Y}\colon\ \mathrm{Perf}(X)\to\mathfrak{D}(\mathrm{QCoh}(Y)),$$

and

$$\mathrm{Hom}(F(\mathcal{O}_X(n)),F(\mathcal{O}_X(m))[j]) = 0 \tag{1.5}$$

for all $j<0$ and all $n,m\in\mathbb{Z}$. Then $\mathcal{E}\cong\mathcal{E}'$.

The above result is an improvement of [1, Theorem 1.1], clearly only regarding the uniqueness problem and the untwisted case: our result holds not only for smooth projective varieties, and under a hypothesis which is weaker than the one in the mentioned article, which is:

$$\mathrm{Hom}(F(\mathcal{F}),F(\mathcal{G})[j]) = 0,$$

for all $\mathcal{F},\mathcal{G}\in\mathrm{Coh}(X)$ and all $j<0$. It is also an improvement of [4, Remark 5.7], which holds for fully faithful functors (in this case, the vanishing condition (1.5) is trivially verified).

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2. Dg-categories and quasi-functors

This section contains the basic well-known facts about dg-categories; we refer to [8] for a comprehensive survey. We fix, once and for all, a ground field $\mathbf{k}$. Virtually every category we shall encounter will be at least $\mathbf{k}$-linear, so we allow ourself some sloppiness, and often use the terms "category" and "functor" meaning "$\mathbf{k}$-category" and "$\mathbf{k}$-functor".

2.1. **Dg-categories and dg-functors.** A dg-category is a category enriched over the closed symmetric monoidal category $\mathsf{C}(\mathbf{k})$ of cochain complexes of $\mathbf{k}$-modules:

Definition 2.1. A *differential graded (dg-) category* $\mathbf{A}$ consists of a set of objects $\operatorname{Ob}\mathbf{A}$, a hom-complex $\mathbf{A}(A, B)$ for any pair of objects A, B, and (unital, associative) composition chain maps of complexes of $\mathbf{k}$-modules:

$$\begin{aligned}\mathbf{A}(B, C) \otimes \mathbf{A}(A, B) &\to \mathbf{A}(A, C),\\ g \otimes f &\mapsto gf = g \circ f\end{aligned}$$

Definition 2.2. Let $\mathbf{A}$ and $\mathbf{B}$ be dg-categories. A *dg-functor* F consists of the following data:

- a map $F\colon \operatorname{Ob}\mathbf{A} \to \operatorname{Ob}\mathbf{B}$;
- for any pair of objects (A, B) of $\mathbf{A}$, a chain map

$$F = F_{(A,B)}\colon \mathbf{A}(A, B) \to \mathbf{B}(F(A), F(B)),$$

subject to the usual associativity and unitality axioms.

Example 2.3. An example of dg-category is given by the *dg-category of complexes* $\mathsf{C}_{\mathrm{dg}}(\mathbf{k})$: it has the same objects as $\mathsf{C}(\mathbf{k})$, and complexes of morphisms $\underline{\operatorname{Hom}}(V, W)$ given by:

$$\begin{aligned}\underline{\operatorname{Hom}}(V, W)^n &= \prod_{i\in\mathbb{Z}} \operatorname{Hom}(V^i, W^{i+k}),\\ df &= d_W \circ f - (-1)^{|f|} f \circ d_V.\end{aligned} \tag{2.1}$$

Remark 2.4. All usual categorical constructions can be carried out for dg-categories and dg-functors.

(1) For any dg-category $\mathbf{A}$ there is the *opposite dg-category* $\mathbf{A}^{\mathrm{op}}$, such that

$$\mathbf{A}^{\mathrm{op}}(A, B) = \mathbf{A}(B, A),$$

with the same compositions as in $\mathbf{A}$ up to a sign:

$$f^{\mathrm{op}} g^{\mathrm{op}} = (-1)^{|f||g|} (gf)^{\mathrm{op}},$$

denoting by $f^{\mathrm{op}} \in \mathbf{A}^{\mathrm{op}}(B, A)$ the morphism corresponding to $f \in \mathbf{A}(A, B)$.

(2) Given dg-categories $\mathbf{A}$ and $\mathbf{B}$, there is the *tensor product* $\mathbf{A} \otimes \mathbf{B}$: its objects are pairs (A, B) where $A \in \mathbf{A}$ and $B \in \mathbf{B}$; its hom-complexes are given by

$$(\mathbf{A} \otimes \mathbf{B})((A, B), (A', B')) = \mathbf{A}(A, A') \otimes \mathbf{B}(B, B').$$

Compositions of two morphisms $f \otimes g$ and $f' \otimes g'$ is given by:

$$(f' \otimes g')(f \otimes g) = (-1)^{|g'||f|} f'f \otimes g'g.$$

The tensor product commutes with taking opposites: $(\mathbf{A} \otimes \mathbf{B})^{\mathrm{op}} = \mathbf{A}^{\mathrm{op}} \otimes \mathbf{B}^{\mathrm{op}}$. Also, it is symmetric, namely, there is an isomorphism of dg-categories: $\mathbf{A} \otimes \mathbf{B} \cong \mathbf{B} \otimes \mathbf{A}$.

(3) Given dg-categories $\mathbf{A}$ and $\mathbf{B}$, there is a dg-category $\operatorname{Fun}_{\mathrm{dg}}(\mathbf{A}, \mathbf{B})$ whose objects are dg-functors $\mathbf{A} \to \mathbf{B}$ and whose complexes of morphisms are the so-called *dg-natural transformations*: A dg-natural transformation $\varphi\colon F \to G$ of degree p is a collection of degree p morphisms

$$\varphi_A\colon F(A) \to G(A),$$

for all $A \in \mathbf{A}$, such that for any degree q morphism $f \in \mathbf{A}(A, A')$ the following diagram is commutative up to the sign $(-1)^{|p||q|}$:

$$\begin{array}{ccc} F(A) & \xrightarrow{\varphi_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(A') & \xrightarrow{\varphi_{A'}} & G(A'). \end{array}$$

Differentials and compositions of dg-natural transformations are defined objectwise.

There is a natural isomorphism of dg-categories:

$$\mathrm{Fun}_{\mathrm{dg}}(\mathbf{A} \otimes \mathbf{B}, \mathbf{C}) \cong \mathrm{Fun}_{\mathrm{dg}}(\mathbf{A}, \mathrm{Fun}_{\mathrm{dg}}(\mathbf{B}, \mathbf{C})). \tag{2.2}$$

Dg-functors $\mathbf{A} \otimes \mathbf{B} \to \mathbf{C}$ are called *dg-bifunctors*, and they are "dg-functors of two variables $A \in \mathbf{A}$ and $B \in \mathbf{B}$", separately dg-functorial in both.

(4) Given a dg-category $\mathbf{A}$, a *(right)* $\mathbf{A}$*-dg-module* is a dg-functor $\mathbf{A}^{\mathrm{op}} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$. We set

$$\mathsf{C}_{\mathrm{dg}}(\mathbf{A}) = \mathrm{Fun}_{\mathrm{dg}}(\mathbf{A}^{\mathrm{op}}, \mathsf{C}_{\mathrm{dg}}(\mathbf{k})).$$

We have a fully faithful dg-functor

$$\begin{aligned} h = h_{\mathbf{A}} \colon \mathbf{A} &\hookrightarrow \mathsf{C}_{\mathrm{dg}}(\mathbf{A}), \\ A &\mapsto \mathbf{A}(-, A), \end{aligned} \tag{2.3}$$

which is the dg version of the Yoneda embedding.

Given dg-categories $\mathbf{A}$ and $\mathbf{B}$, an $\mathbf{A}$-$\mathbf{B}$-*dg-bimodule* (covariant in $\mathbf{A}$, contravariant in $\mathbf{B}$) is a right $\mathbf{B} \otimes \mathbf{A}^{\mathrm{op}}$-dg-module, namely a dg-functor $\mathbf{B}^{\mathrm{op}} \otimes \mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{k})$. By convention, the contravariant variable comes first. By (2.2), such a bimodule can also be viewed as a dg-functor $\mathbf{A} \to \mathsf{C}_{\mathrm{dg}}(\mathbf{B})$.

Remark 2.5. Any ordinary ($\mathbf{k}$-linear) category can be viewed as a dg-category, with complexes of morphisms concentrated in degree zero.

2.2. **The derived category.** Small dg-categories and dg-functors form a category, which is denoted by $\mathsf{dgCat}_{\mathbf{k}}$, or simply dgCat when the base field is clear. The operations of taking cocycles and cohomology can be extended from complexes of $\mathbf{k}$-modules to dg-categories and dg-functors:

Definition 2.6. Let $\mathbf{A}$ be a dg-category. The *underlying category* (resp. the *homotopy category*) of $\mathbf{A}$ is the category $Z^0(\mathbf{A})$ (resp. $H^0(\mathbf{A})$) which is defined as follows:

- $\mathrm{Ob}\, Z^0(\mathbf{A}) = \mathrm{Ob}\, H^0(\mathbf{A}) = \mathrm{Ob}\, \mathbf{A}$,
- $Z^0(\mathbf{A})(A, B) = Z^0(\mathbf{A}(A, B))$ (respectively $H^0(\mathbf{A})(A, B) = H^0(\mathbf{A}(A, B))$), for all $A, B \in \mathbf{A}$,

with natural compositions and identities.

The mappings $\mathbf{A} \mapsto Z^0(\mathbf{A})$ and $\mathbf{A} \mapsto H^0(\mathbf{A})$ yield functors $H^0, Z^0 \colon \mathsf{dgCat} \to \mathsf{Cat}$, where $\mathsf{Cat} = \mathsf{Cat}_{\mathbf{k}}$ is the category of (small, $\mathbf{k}$-linear) categories: given a dg-functor $F \colon \mathbf{A} \to \mathbf{B}$, there are natural induced functors

$$\begin{aligned} Z^0(F) \colon Z^0(\mathbf{A}) \to Z^0(\mathbf{B}), \\ H^0(F) \colon H^0(\mathbf{A}) \to H^0(\mathbf{B}). \end{aligned}$$

Given two objects A, B in a dg-category $\mathbf{A}$, we say that they are *dg-isomorphic* (resp. *homotopy equivalent*), and write $A \cong B$ (resp. $A \approx B$) if they are isomorphic in $Z^0(\mathbf{A})$ (resp. $H^0(\mathbf{A})$).

Let $\mathbf{A}$ be a dg-category. The *homotopy category of* $\mathbf{A}$*-modules* is defined to be $\mathsf{K}(\mathbf{A}) = H^0(\mathsf{C}_{\mathrm{dg}}(\mathbf{A}))$. A morphism $M \to N$ in $\mathsf{K}(\mathbf{A})$ is a *quasi-isomorphism* if $M(A) \to N(A)$ is a quasi-isomorphism of complexes for all $A \in \mathbf{A}$. The *derived category* of $\mathbf{A}$ is defined to be the localisation of $\mathsf{K}(\mathbf{A})$ along quasi-isomorphisms:

$$\mathsf{D}(\mathbf{A}) = \mathsf{K}(\mathbf{A})[\mathrm{Qis}^{-1}].$$

The Yoneda embedding induces a fully faithful functor:

$$H^0(\mathbf{A}) \hookrightarrow \mathsf{D}(\mathbf{A}), \tag{2.4}$$

which is called the *derived Yoneda embedding*. The category $\mathsf{D}(\mathbf{A})$ is triangulated; as in the case of the derived category of complexes of $\mathbf{k}$-modules, morphisms $T \to T'$ in $\mathsf{D}(\mathbf{A})$ are represented by "roofs"

$$T \xleftarrow{\approx} T'' \to T',$$

in $\mathsf{K}(\mathbf{A})$, where the arrow $T'' \to T$ is a quasi-isomorphism. Two $\mathbf{A}$-dg-modules T and T' are *quasi-isomorphic* ($T \overset{\mathrm{qis}}{\approx} T'$) if they are isomorphic in $\mathsf{D}(\mathbf{A})$, which is equivalent to saying that there is a "roof" of quasi-isomorphisms:

$$T \xleftarrow{\approx} T'' \xrightarrow{\approx} T'. \tag{2.5}$$

We denote by $\mathrm{tria}(\mathbf{A})$ the smallest strictly full triangulated subcategory of $\mathsf{D}(\mathbf{A})$ which contains the image of (2.4). Moreover, we denote by $\mathrm{per}(\mathbf{A})$ the idempotent completion of $\mathrm{tria}(\mathbf{A})$, which coincides with the smallest strictly full triangulated subcategory of $\mathsf{D}(\mathbf{A})$ which contains the image of (2.4) and is thick, i.e. closed under direct summands; it can also be characterised as the subcategory of compact objects in $\mathsf{D}(\mathbf{A})$. The derived Yoneda embedding factors through $\mathrm{tria}(\mathbf{A})$:

$$H^0(\mathbf{A}) \hookrightarrow \mathrm{tria}(\mathbf{A}) \hookrightarrow \mathrm{per}(\mathbf{A}). \tag{2.6}$$

Definition 2.7. A dg-category $\mathbf{A}$ is *pretriangulated* if $H^0(\mathbf{A}) \hookrightarrow \mathrm{tria}(\mathbf{A})$ is an equivalence; it is *triangulated* if $H^0(\mathbf{A}) \hookrightarrow \mathrm{per}(\mathbf{A})$ is an equivalence.

We remark that a pretriangulated dg-category $\mathbf{A}$ is triangulated if and only if $H^0(\mathbf{A})$ is idempotent complete.

Pretriangulated dg-categories are used as higher categorical models for triangulated categories. A *dg-enhancement* of a triangulated category $\mathbf{T}$ is a pretriangulated dg-category $\mathbf{A}$ together with an equivalence $H^0(\mathbf{A}) \xrightarrow{\sim} \mathbf{T}$; we will often forget about this equivalence and just say that $\mathbf{A}$ is a dg-enhancement of $\mathbf{T}$.

2.3. **Quasi-functors.** The category dgCat carries significant homotopical structure. A *quasi-equivalence* is a dg-functor $F\colon \mathbf{A} \to \mathbf{B}$ such that the maps

$$F_{(A,B)}\colon \mathbf{A}(A,B) \to \mathbf{B}(F(A), F(B))$$

are quasi-isomorphisms, and $H^0(F)$ is essentially surjective. Given dg-categories $\mathbf{A}$ and $\mathbf{B}$, we say that they are *quasi-equivalent* if there exists a zig-zag of quasi-equivalences:

$$\mathbf{A} \leftarrow \mathbf{A}_1 \to \ldots \leftarrow \mathbf{A}_n \to \mathbf{B}.$$

Model category theory allows us to understand dg-categories up to quasi-equivalence. We summarise the main results in the following statement:

Theorem 2.8 (([15], [17]))**.** *The category* dgCat *of small dg-categories has a model category structure whose weak equivalences are the quasi-equivalences; the localisation of* dgCat *along quasi-equivalences is denoted by* Hqe*; two dg-categories are quasi-equivalent if and only if they are isomorphic in* Hqe*. Given dg-categories* $\mathbf{A}$ *and* $\mathbf{B}$*, there exists a dg-category* $\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A},\mathbf{B})$ *which is the internal hom between* $\mathbf{A}$ *and* $\mathbf{B}$ *in* Hqe*, namely there is a natural bijection:*

$$\mathsf{Hqe}(\mathbf{A} \otimes \mathbf{B}, \mathbf{C}) \xrightarrow{\sim} \mathsf{Hqe}(\mathbf{A}, \mathbb{R}\underline{\mathrm{Hom}}(\mathbf{B},\mathbf{C})). \tag{2.7}$$

Furthermore, there is a natural bijection

$$\mathsf{Hqe}(\mathbf{A},\mathbf{B}) \xrightarrow{\sim} \mathrm{Iso}(H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A},\mathbf{B}))) \tag{2.8}$$

between the set of morphisms $\mathbf{A} \to \mathbf{B}$ *in* Hqe *and the set of isomorphism classes of objects in* $H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A},\mathbf{B}))$.

The internal hom $\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A},\mathbf{B})$ is defined up to isomorphism in Hqe and yields a functor

$$\mathbb{R}\underline{\mathrm{Hom}}(-,-)\colon \mathsf{Hqe}^{\mathrm{op}} \times \mathsf{Hqe} \to \mathsf{Hqe}.$$

Moreover, the bijection (2.7) lifts to a natural isomorphism in Hqe:

$$\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}\otimes\mathbf{B},\mathbf{C}) \cong \mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A},\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{B},\mathbf{C})).$$

Objects of $\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A},\mathbf{B})$ are called *quasi-functors*: they are the "homotopy relevant" functors between dg-categories. Quasi-functors can be described concretely as particular bimodules:

Proposition 2.9 (([8, Theorem 4.5]))**.** *The category* $H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A},\mathbf{B}))$ *is equivalent to the category* $\mathrm{qrep}^r(\mathbf{B}\otimes\mathbf{A}^{\mathrm{op}})$ *of* right quasi-representable $\mathbf{A}$-$\mathbf{B}$*-dg-bimodules, namely the full subcategory of* $\mathsf{D}(\mathbf{B}\otimes\mathbf{A}^{\mathrm{op}})$ *of* $\mathbf{A}$-$\mathbf{B}$*-dg-bimodules* T *such that for all* $A \in \mathbf{A}$ *the* $\mathbf{B}$*-module* $T(-,A)$ *is quasi-isomorphic to* $\mathbf{B}(-,B')$ *for some* $B' \in \mathbf{B}$.

2.4. **Pretriangulated hulls.** Let $\mathbf{A}$ be a dg-category. A dg-module $X \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A})$ is *acyclic* if $X(A)$ is an acyclic complex for all $A \in \mathbf{A}$. A dg-module $M \in \mathsf{C}_{\mathrm{dg}}(\mathbf{A})$ is *h-projective* if $\mathsf{K}(\mathbf{A})(M,X) = 0$ for any acyclic $\mathbf{A}$-dg-module X. H-projective dg-modules serve as an enhancement of the derived category:

Proposition 2.10. *The full dg-subcategory* $\text{h-proj}(\mathbf{A}) \subset \mathsf{C}_{\mathrm{dg}}(\mathbf{A})$ *of h-projective* $\mathbf{A}$*-dg-modules is an enhancement of* $\mathsf{D}(\mathbf{A})$*. Namely, the composition*

$$H^0(\text{h-proj}(\mathbf{A})) \hookrightarrow \mathsf{K}(\mathbf{A}) \to \mathsf{D}(\mathbf{A})$$

is an equivalence.

Taking suitable dg-subcategories of $\text{h-proj}(\mathbf{A})$, we obtain enhancements $\mathrm{pretr}(\mathbf{A})$ and $\mathrm{per}_{\mathrm{dg}}(\mathbf{A})$ of $\mathrm{tria}(\mathbf{A})$ and $\mathrm{per}(\mathbf{A})$, respectively. For instance, $\mathrm{per}_{\mathrm{dg}}(\mathbf{A})$ is defined as the full dg-subcategory of $\text{h-proj}(\mathbf{A})$ whose objects are in $\mathrm{per}(\mathbf{A})$. The dg-Yoneda embedding factors through $\mathrm{pretr}(\mathbf{A})$:

$$\mathbf{A} \hookrightarrow \mathrm{pretr}(\mathbf{A}) \hookrightarrow \mathrm{per}_{\mathrm{dg}}(\mathbf{A}). \tag{2.9}$$

The dg-category $\mathrm{per}_{\mathrm{dg}}(\mathbf{A})$ is called the *triangulated hull* of $\mathbf{A}$. It satisfies the following "homotopy universal property":

Proposition 2.11 (([18, Proposition 5.1.2], [17, Theorem 7.2], [5, Corollary 4.2]))**.** *Let* $\mathbf{A},\mathbf{B}$ *be dg-categories, and assume that* $\mathbf{B}$ *is triangulated. Then* $\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A},\mathbf{B})$ *is triangulated. Moreover, there is a natural isomorphism in* Hqe*:*

$$\mathbb{R}\underline{\mathrm{Hom}}(\mathrm{per}_{\mathrm{dg}}(\mathbf{A}),\mathbf{B}) \xrightarrow{\sim} \mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A},\mathbf{B}), \tag{2.10}$$

induced by the Yoneda embedding $\mathbf{A} \hookrightarrow \mathrm{per}_{\mathrm{dg}}(\mathbf{A})$.

3. The uniqueness problem of dg-lifts

Any quasi-functor $T\colon \mathbf{A} \to \mathbf{B}$ yields an ordinary functor $H^0(T)\colon H^0(\mathbf{A}) \to H^0(\mathbf{B})$. More precisely, there is a functor:

$$\Phi^{\mathbf{A}\to\mathbf{B}}\colon H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A},\mathbf{B})) \to \mathrm{Fun}(H^0(\mathbf{A}),H^0(\mathbf{B})). \tag{3.1}$$

When we identify $H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A},\mathbf{B}))$ with $\mathrm{qrep}^r(\mathbf{B}\otimes\mathbf{A}^{\mathrm{op}})$, this is described as follows. Let T be an $\mathbf{A}$-$\mathbf{B}$-dg-bimodule, and view it as a dg-functor

$$\begin{aligned} \mathbf{A} &\to \mathsf{C}_{\mathrm{dg}}(\mathbf{B}), \\ A &\mapsto T(-,A). \end{aligned}$$

This induces an ordinary functor

$$H^0(\mathbf{A}) \to \mathsf{K}(\mathbf{B}) \to \mathsf{D}(\mathbf{B}).$$

Since T is right quasi-representable, this functor factors through the derived Yoneda embedding of $\mathbf{B}$, yielding $H^0(T) = \Phi^{\mathbf{A}\to\mathbf{B}}(T)$:

$$H^0(\mathbf{A}) \xrightarrow{H^0(T)} H^0(\mathbf{B}) \hookrightarrow \mathsf{D}(\mathbf{B}).$$

Remark 3.1. The functor $H^0\colon \mathsf{dgCat} \to \mathsf{Cat}$ maps quasi-equivalences to equivalences, hence it induces a functor

$$[H^0(-)]\colon \mathsf{Hqe} \to [\mathsf{Cat}],$$

where $[\mathsf{Cat}]$ is the category of (small, $\mathbf{k}$-linear) categories modulo equivalences. If we identify $\mathsf{Hqe}(\mathbf{A},\mathbf{B})$ with $\mathrm{Iso}(H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A},\mathbf{B})))$, then the map

$$[H^0(-)]\colon \mathsf{Hqe}(\mathbf{A},\mathbf{B}) \to [\mathsf{Cat}](H^0(\mathbf{A}),H^0(\mathbf{B})) \tag{3.2}$$

is precisely

$$\mathrm{Iso}(\Phi^{\mathbf{A}\to\mathbf{B}})\colon \mathrm{Iso}(H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A},\mathbf{B}))) \to \mathrm{Iso}(\mathrm{Fun}(H^0(\mathbf{A}),H^0(\mathbf{B})))$$

(see also [17, proof of Lemma 4.3]). This gives another explanation why notation H^0 is used for $\Phi^{\mathbf{A}\to\mathbf{B}}$.

If $\mathbf{A}$ and $\mathbf{B}$ are pretriangulated, then $\Phi^{\mathbf{A}\to\mathbf{B}}$ takes values in the category of exact functors $\mathrm{Fun}_{\mathrm{ex}}(H^0(\mathbf{A}),H^0(\mathbf{B}))$, and a similar fact holds for (3.2). The *uniqueness problem of dg-lifts* amounts to understanding in which cases $\Phi^{\mathbf{A}\to\mathbf{B}}$ is essentially injective (or, equivalently, (3.2) is injective), that is: given quasi-functors T_1, T_2 such that $H^0(T_1) \cong H^0(T_2)$, is it true that $T_1 \cong T_2$ in $H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A},\mathbf{B}))$? In many situations, we will be studying dg-functors whose domain dg-category $\mathbf{A}$ is (pre)triangulated and generated by a simpler dg-category, namely $\mathbf{A}$ is quasi-equivalent to a dg-category of the form $\mathrm{per}_{\mathrm{dg}}(\mathbf{C})$. In this case, the uniqueness problem of dg-lifts can be reduced to generators:

Lemma 3.2. *Let $\mathbf{A}$ and $\mathbf{B}$ be triangulated dg-categories, and assume that $\mathbf{A}$ is quasi-equivalent to $\mathrm{per}_{\mathrm{dg}}(\mathbf{C})$ for some dg-category $\mathbf{C}$. Then, $\Phi^{\mathbf{A}\to\mathbf{B}}$ is essentially injective if $\Phi^{\mathbf{C}\to\mathbf{B}}$ is such.*

Proof. Without loss of generality, we may identify $\mathbf{A} = \mathrm{per}_{\mathrm{dg}}(\mathbf{C})$. There is a commutative diagram of categories and functors:

$$\begin{array}{ccc} H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathrm{per}_{\mathrm{dg}}(\mathbf{C}),\mathbf{B})) & \xrightarrow{\Phi^{\mathbf{A}\to\mathbf{B}}} & \mathrm{Fun}_{\mathrm{ex}}(H^0(\mathrm{per}_{\mathrm{dg}}(\mathbf{C})),H^0(\mathbf{B})) \\ \downarrow{\sim} & & \downarrow \\ H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{C},\mathbf{B})) & \xrightarrow{\Phi^{\mathbf{C}\to\mathbf{B}}} & \mathrm{Fun}(H^0(\mathbf{C}),H^0(\mathbf{B})), \end{array}$$

where the left vertical arrow is induced by the Yoneda embedding $\mathbf{C} \hookrightarrow \mathrm{per}_{\mathrm{dg}}(\mathbf{C})$, and the right vertical arrow is induced by its zeroth cohomology: $H^0(\mathbf{C}) \hookrightarrow H^0(\mathrm{per}_{\mathrm{dg}}(\mathbf{C}))$. By Proposition 2.11, the left vertical arrow is an equivalence; the claim now follows from a direct argument. □

We mention another relevant property of $\Phi^{\mathbf{A}\to\mathbf{B}}$:

Proposition 3.3. *The functor* (3.1) *reflects isomorphisms.*

Proof. A morphism $T \to T'$ in $H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A},\mathbf{B})) = \mathrm{qrep}^r(\mathbf{A},\mathbf{B})$ is given by a roof

$$T \xleftarrow{\approx} T'' \to T'$$

in $\mathsf{K}(\mathbf{B} \otimes \mathbf{A}^{\mathrm{op}})$, where the arrow $T'' \to T$ is a quasi-isomorphism. So, it sufficient to prove that any morphism of $\mathbf{A}$-$\mathbf{B}$-bimodules $\varphi\colon T \to T'$ is a quasi-isomorphism if $H^0(\varphi)\colon H^0(T) \to H^0(T')$ is an isomorphism. Now, φ is a quasi-isomorphism if and only if $\varphi_A : T(-,A) \to T'(-,A)$ is an isomorphism in $\mathsf{D}(\mathbf{B})$ for all $A \in \mathbf{A}$, which is equivalent to requiring that $\varphi'_A\colon \mathbf{B}(-,B_A) \to \mathbf{B}(-,B'_A)$ is an isomorphism in $\mathsf{D}(\mathbf{B})$ for all A, where φ'_A is the unique morphism in $\mathsf{D}(\mathbf{B})$ such that the following diagram is commutative in $\mathsf{D}(\mathbf{B})$:

$$\begin{array}{ccc} T(-,A) & \xrightarrow{\varphi_A} & T'(-,A) \\ \downarrow \approx & & \downarrow \approx \\ \mathbf{B}(-,B_A) & \xrightarrow{\varphi'_A} & \mathbf{B}(-,B'_A) \end{array}$$

(the vertical arrows are chosen isomorphisms for all $A \in \mathbf{A}$). By definition, $H^0(\varphi)_A$ is the map $B_A \to B'_A$ in $H^0(\mathbf{B})$ whose image under the derived Yoneda embedding $H^0(\mathbf{B}) \hookrightarrow \mathsf{D}(\mathbf{B})$ is precisely φ'_A. So, if $H^0(\varphi)_A$ is an isomorphism in $H^0(\mathbf{B})$ for all $A \in \mathbf{A}$, then φ'_A is an isomorphism in $\mathsf{D}(\mathbf{B})$ for all A, and we are done. ☐

4. Dg-lifts and A_∞-functors

A_∞-*categories* and A_∞-functors are, respectively, a homotopy coherent incarnation of dg-categories and dg-functors. A_∞-functors are actually a model for quasi-functors; their advantage over quasi-representable bimodules relies in their "concreteness": they are defined by elementary (even if quite complicated) formulae, which can be used in rather direct arguments. This formalism will allow us to prove a uniqueness result for dg-lifts under some hypothesis on the functors involved

4.1. **A_∞-categories and functors.** The basic notions of the theory of A_∞-categories and functors are taken directly from [14], whose conventions will be followed. We warn the reader especially about sign conventions, which are possibly the most annoying feature of the theory. If it feels more comfortable, just assume that $\mathrm{char}\,\mathbf{k} = 2$, at least at a first reading.

We will be working with *strictly unital* A_∞-categories and functors. The formal definitions are as follows:

Definition 4.1. A *strictly unital* A_∞-category $\mathbf{A}$ consists of a set of objects $\mathrm{Ob}\,\mathbf{A}$, a graded $\mathbf{k}$-vector space $\mathbf{A}(X_0,X_1)$ for any pair of objects $X_0, X_1 \in \mathbf{A}$, and (degree 0) maps of graded vector spaces for any order $d \geq 1$:

$$\mu^d_{\mathbf{A}}\colon \mathbf{A}(X_{d-1},X_d) \otimes \ldots \otimes \mathbf{A}(X_0,X_1) \to \mathbf{A}(X_0,X_d)[2-d], \tag{4.1}$$

satisfying the following collection of equations (for all $d \geq 1$):

$$\sum_{m=1}^{d}\sum_{n=0}^{d-m}(-1)^{\maltese_n}\mu^{d-m+1}_{\mathbf{A}}(f_d,\ldots,f_{n+m+1},\mu^m_{\mathbf{A}}(f_{n+m},\ldots,f_{n+1}),f_n,\ldots,f_1) = 0, \tag{4.2}$$

where by definition $\maltese_n = |f_1| + \ldots + |f_n| - n$. Moreover, for any object $X \in \mathbf{A}$, there exists a (necessarily unique) morphism $1_X \in \mathbf{A}(X,X)^0$ which satisfies:

$$\begin{aligned} &\mu^1_{\mathbf{A}}(1_X) = 0, \\ &(-1)^{|f|}\mu^2_{\mathbf{A}}(1_{X_1}, f) = \mu^2_{\mathbf{A}}(f, 1_{X_0}) = f, \quad \forall\, f \in \mathbf{A}(X_0, X_1), \\ &\mu^d_{\mathbf{A}}(f_{d-1}, \ldots, f_{n+1}, 1_{X_n}, f_n, \ldots, f_1) = 0, \\ &\forall\, d > 2, f_k \in \mathbf{A}(X_{k-1}, X_k), \forall\, 0 \leq n < d. \end{aligned} \tag{4.3}$$

Unwinding the above definition, we find out that that the map $\mu^1_{\mathbf{A}}$ is a differential which endows the hom-spaces $\mathbf{A}(X,Y)$ with a structure of chain complex. The composition $\mu^2_{\mathbf{A}}$ is not associative, but its deviation from being so is measured by the higher compositions $\mu^d_{\mathbf{A}}$ for $d \geq 3$.

Definition 4.2. Let $\mathbf{A}$ and $\mathbf{B}$ be (strictly unital) A_∞-categories. An A_∞-functor $F\colon \mathbf{A} \to \mathbf{B}$ consists of a map of sets

$$\begin{aligned} F^0\colon\ &\operatorname{Ob}\mathbf{A} \to \operatorname{Ob}\mathbf{B}, \\ &X \mapsto F^0(X) = F(X), \end{aligned}$$

and (degree 0) maps of graded vector spaces

$$F^d\colon \mathbf{A}(X_{d-1}, X_d) \otimes \ldots \otimes \mathbf{A}(X_0, X_1) \to \mathbf{B}(F(X_0), F(X_d))[1-d], \tag{4.4}$$

subject to the following equations, for all $d \geq 1$:

$$\begin{aligned} &\sum_{r \geq 1} \sum_{s_1 + \ldots + s_r = d} \mu^r_{\mathbf{B}}(F^{s_r}(f_d, \ldots, f_{d-s_r+1}), \ldots, F^{s_1}(f_{s_1}, \ldots, f_1)) \\ &= \sum_{m=1}^{d} \sum_{n=0}^{d-m} (-1)^{\maltese_n} F^{d-m+1}(f_d, \ldots, f_{n+m+1}, \mu^m_{\mathbf{A}}(f_{n+m}, \ldots, f_{n+1}), f_n, \ldots, f_1), \end{aligned} \tag{4.5}$$

where $s_i \geq 1$ for all i. Moreover, F is required to satisfy the following strict unitality condition:

$$\begin{aligned} &F^1(1_X) = 1_{F(X)}, \quad \forall\, X \in \mathbf{A}, \\ &F^d(f_{d-1}, \ldots, f_{n+1}, 1_{X_n}, f_n, \ldots, f_1) = 0, \\ &\forall\, d \geq 2, f_k \in \mathbf{A}(X_{k-1}, X_k), \forall\, 0 \leq n < d. \end{aligned} \tag{4.6}$$

Given A_∞-functors $F\colon \mathbf{A} \to \mathbf{B}$ and $G\colon \mathbf{B} \to \mathbf{C}$, their *composition* $G \circ F$ is defined as follows:

$$\begin{aligned} &(G \circ F)^0 = G^0 \circ F^0, \\ &(G \circ F)^d(f_d, \ldots, f_1) \\ &\qquad = \sum_{r \geq 1} \sum_{s_1 + \ldots + s_r = d} G^r(F^{s_r}(f_d, \ldots, f_{d-s_r+1}), \ldots, F^{s_1}(f_{s_1}, \ldots, f_1)), \end{aligned} \tag{4.7}$$

whenever $d \geq 1$, with $s_i \geq 1$.

Remark 4.3. Any dg-category $\mathbf{A}$ can be viewed as an A_∞-category, setting

$$\begin{aligned} \mu^1_{\mathbf{A}}(f) &= (-1)^{|f|} df, \\ \mu^2_{\mathbf{A}}(g, f) &= (-1)^{|f|} gf, \\ \mu^d_{\mathbf{A}} &= 0, \quad \forall\, d > 2. \end{aligned}$$

As we see, apart from sign twists, a dg-category is an A_∞-category whose higher compositions (for $d > 2$) vanish. From now on, unless otherwise specified, any dg-category will be implicitly viewed in this way as an A_∞-category.

It is interesting to see how the definition of A_∞-functor behaves if the domain and codomain are assumed to be dg-categories. If $F\colon \mathbf{A} \to \mathbf{B}$ is an A_∞-functor between dg-categories, the degree d equation (4.5) boils down to:

$$\begin{aligned} &\mu^1_{\mathbf{B}}(F^d(f_d,\dots,f_1)) + \sum_{j=1}^{d-1} \mu^2_{\mathbf{B}}(F^j(f_d,\dots,f_{d-j+1}), F^{d-j}(f_{d-j},\dots,f_1)) \\ &= \sum_{n=0}^{d-1} (-1)^{\maltese_n} F^d(f_d,\dots,f_{n+2},\mu^1_{\mathbf{A}}(f_{n+1}),f_n,\dots,f_1) \\ &\quad + \sum_{n=0}^{d-2} (-1)^{\maltese_n} F^{d-1}(f_d,\dots,f_{n+3},\mu^2_{\mathbf{A}}(f_{n+2},f_{n+1}),f_n,\dots,f_1). \end{aligned} \tag{4.8}$$

In the even simpler case when $F\colon \mathbb{E} \to \mathbf{B}$ is an A_∞-functor where $\mathbb{E}$ is a $\mathbf{k}$-linear category ($\mu^1_{\mathbb{E}} = 0$, $\mu^2_{\mathbb{E}}$ is the composition, $\mu^d_{\mathbb{E}} = 0$ for $d > 2$) and $\mathbf{B}$ is a dg-category, the degree d equation defining F then reduces to the following:

$$\begin{aligned} &\mu^1_{\mathbf{B}}(F^d(f_d,\dots,f_1)) + \sum_{j=1}^{d-1} \mu^2_{\mathbf{B}}(F^j(f_d,\dots,f_{d-j+1}), F^{d-j}(f_{d-j},\dots,f_1)) \\ &\qquad = \sum_{n=0}^{d-2} (-1)^{\maltese_n} F^{d-1}(f_d,\dots f_{n+3},\mu^2_{\mathbb{E}}(f_{n+2},f_{n+1}),f_n,\dots,f_1). \end{aligned} \tag{4.9}$$

It is also interesting to see what is the composition of an A_∞-functor $F\colon \mathbf{A} \to \mathbf{B}$ (between dg-categories) with a dg-functor $G\colon \mathbf{B} \to \mathbf{C}$. Such a dg-functor, viewed as an A_∞-functor, is characterised by having $G^d = 0$ for all $d > 1$. Formula (4.7) becomes very simple:

$$(G \circ F)^d(f_d,\dots,f_1) = G^1(F^d(f_d,\dots,f_1)), \tag{4.10}$$

for all $d \geq 1$.

Given A_∞-categories $\mathbf{A}$ and $\mathbf{B}$, there is an *A_∞-category* $\mathrm{Fun}_\infty(\mathbf{A},\mathbf{B})$ *of (strictly unital) A_∞-functors*. Its definition involves describing *(A_∞-)natural transformations* of A_∞-functors.

Definition 4.4. Let $F, G\colon \mathbf{A} \to \mathbf{B}$ be A_∞-functors. A *degree g pre-natural transformation* $h\colon F \to G$ is consists of a sequence of maps $(h^0, h^1, \dots)$ such that

$$h^0\colon X \mapsto h^0_X \in \mathbf{B}(F(X), G(X))^g, \quad X \in \mathbf{A},$$

and h^d, for $d \geq 1$, is a family of (degree 0) maps of graded vector spaces

$$h^d\colon \mathbf{A}(X_{d-1}, X_d) \otimes \ldots \otimes \mathbf{A}(X_0, X_1) \to \mathbf{B}(F(X_0), G(X_d))[g-d]$$

for any family of objects $X_0, \dots, X_d \in \mathbf{A}$. Moreover, we require the strict unitality condition:

$$h^d(f_{d-1},\dots,f_{n+1},1_{X_n},f_n,\dots,f_1) = 0, \tag{4.11}$$

for all $d \geq 1$ and $0 \leq n < d$, with $f_k \in \mathbf{A}(X_{k-1}, X_k)$.

Pre-natural transformations $F \to G$ form the graded vector space $\mathrm{Fun}_\infty(\mathbf{A},\mathbf{B})(F,G)$. (Higher) Compositions μ^d are described in [14, Paragraph (1d)]. For example, we have that

$$\mu^1(h)^0_X = \mu^1_{\mathbf{B}}(h^0_X), \quad \forall\, X \in \mathbf{A}.$$

This defines the A_∞-category $\mathrm{Fun}_\infty(\mathbf{A},\mathbf{B})$ of (strictly unital) A_∞-functors.

Remark 4.5. It is worth writing down the coboundary formula for a pre-natural transformation $h\colon F \to G$ if $F, G\colon \mathbf{A} \to \mathbf{B}$ are A_∞-functors between dg-categories. If $d \geq 1$, we have:

$$\mu^1(h)^d(f_d,\dots,f_1) = A^d - B^d, \tag{4.12}$$

where

$$\begin{aligned} A^d =& \mu^1_{\mathbf{B}}(h^d(f_d,\dots,f_1)) \\ &+ \mu^2_{\mathbf{B}}(G^d(f_d,\dots,f_1),h^0_{X_0}) + (-1)^{\maltese_d(|h|-1)}\mu^2_{\mathbf{B}}(h^0_{X_d},F^d(f_d,\dots,f_1)) \\ &+ \sum_{j=1}^{d-1}\mu^2_{\mathbf{B}}(G^j(f_d,\dots,f_{d-j+1}),h^{d-j}(f_{d-j},\dots,f_1)) \\ &+ \sum_{j=1}^{d-1}(-1)^{\maltese_{d-j}(|h|-1)}\mu^2_{\mathbf{B}}(h^j(f_d,\dots,f_{d-j+1}),F^{d-k}(f_{d-j},\dots,f_1)), \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} B^d =& \sum_{n=0}^{d-1}(-1)^{\maltese_n+|h|-1}h^d(f_d,\dots,f_{n+2},\mu^1_{\mathbf{A}}(f_{n+1}),f_n,\dots,f_1) \\ &+ \sum_{n=0}^{d-2}(-1)^{\maltese_n+|h|-1}h^{d-1}(f_d,\dots,f_{n+3},\mu^2_{\mathbf{A}}(f_{n+2},f_{n+1}),f_n,\dots,f_1), \end{aligned} \tag{4.14}$$

given composable morphisms $f_1,\dots,f_d$ with first source X_0 and final target X_d. Notice that the term B_d is similar to the right hand side of (4.8).

4.2. **Natural transformations.** Closed degree 0 pre-natural transformations of A_∞-functors are called *natural transformations*. We are going to describe a useful characterisation of them; we start with a definition in the dg-setting:

Definition 4.6. Let $\mathbf{A}$ be a dg-category (here, *not* viewed as an A_∞-category). The *dg-category of (homotopy coherent) morphisms* $\underline{\operatorname{Mor}}\,\mathbf{A}$ is defined as follows. Objects are triples (A,B,f), where $f\in Z^0(\mathbf{A}(A,B))$. A degree n morphism $(A,B,f)\to(A',B',f')$ is given by a lower triangular matrix

$$(u,v,h)=\begin{pmatrix} u & 0 \\ h & v\end{pmatrix},$$

where $u\in\mathbf{A}(A,A')^n$, $v\in\mathbf{A}(B,B')^n$ and $h\in\mathbf{A}(A,B')^{n-1}$. Compositions are defined by matrix multiplication with a sign rule:

$$\begin{pmatrix} u' & 0 \\ h' & v'\end{pmatrix}\begin{pmatrix} u & 0 \\ h & v\end{pmatrix}=\begin{pmatrix} u'u & 0 \\ (-1)^n h'u+v'h & v'v\end{pmatrix},$$

whenever (u,v,h) has degree n. The differential of a morphism $(u,v,h)\colon(A,B,f)\to(A',B',f')$ of degree n is defined by

$$d\begin{pmatrix} u & 0 \\ h & v\end{pmatrix}=\begin{pmatrix} du & 0 \\ dh+(-1)^n(f'u-vf) & dv\end{pmatrix}.$$

There are obvious "source" and "target" dg-functors:

$$\begin{aligned} S\colon \underline{\operatorname{Mor}}\,\mathbf{A}\to\mathbf{A},\quad (A,B,f)\mapsto A,\quad (u,v,h)\mapsto u,\\ T\colon \underline{\operatorname{Mor}}\,\mathbf{A}\to\mathbf{A},\quad (A,B,f)\mapsto B,\quad (u,v,h)\mapsto v.\end{aligned}$$

Notice that the chosen sign conventions in the definition of $\underline{\operatorname{Mor}}\,\mathbf{A}$ (the same as in [5, 2.2]) allow to define S and T in the simplest way, without any sign twist.

We remark that there is a natural functor

$$\begin{aligned} H^0(\underline{\operatorname{Mor}}\,\mathbf{A}) &\to \operatorname{Mor} H^0(\mathbf{A}),\\ (A,B,f) &\mapsto (A,B,[f]),\\ [(u,v,h)] &\mapsto ([u],[v]), \end{aligned} \tag{4.15}$$

where $\operatorname{Mor} H^0(\mathbf{A})$ denotes the ordinary category of morphisms of $H^0(\mathbf{A})$.

Example 4.7. Let $\mathbf{A}$ be a dg-category, now viewed as an A_∞-category (Remark 4.3). Let us write down what happens when we view the dg-category of homotopy coherent morphisms $\mathbf{Q} = \underline{\operatorname{Mor}}\, \mathbf{A}$ as an A_∞-category. First:

$$\begin{aligned}
\mu^1_{\mathbf{Q}} \begin{pmatrix} u & 0 \\ h & v \end{pmatrix} &= (-1)^{|u|} \begin{pmatrix} du & 0 \\ dh + (-1)^{|u|}(f'u - vf) & dv \end{pmatrix} \\
&= (-1)^{|u|} \begin{pmatrix} (-1)^{|u|}\mu^1_{\mathbf{A}}(u) & 0 \\ (-1)^{|u|-1}\mu^1_{\mathbf{A}}(h) + (-1)^{|u|}((-1)^{|u|}\mu^2_{\mathbf{A}}(f', u) - \mu^2_{\mathbf{A}}(v, f)) & (-1)^{|u|}\mu^1_{\mathbf{A}}(v) \end{pmatrix} \\
&= \begin{pmatrix} \mu^1_{\mathbf{A}}(u) & 0 \\ -\mu^1_{\mathbf{A}}(h) + (-1)^{|u|}\mu^2_{\mathbf{A}}(f', u) - \mu^2_{\mathbf{A}}(v, f) & \mu^1_{\mathbf{A}}(v) \end{pmatrix}.
\end{aligned}$$

Moreover:

$$\begin{aligned}
&\mu^2_{\mathbf{Q}} \left(\begin{pmatrix} u' & 0 \\ h' & v' \end{pmatrix}, \begin{pmatrix} u & 0 \\ h & v \end{pmatrix} \right) \\
&\quad = \begin{pmatrix} (-1)^{|u|}u'u & 0 \\ (-1)^{|u|}((-1)^{|u|}h'u + v'h) & (-1)^{|u|}v'v \end{pmatrix} \\
&\quad = \begin{pmatrix} \mu^2_{\mathbf{A}}(u', u) & 0 \\ (-1)^{|u|}\mu^2_{\mathbf{A}}(h', u) - \mu^2_{\mathbf{A}}(v', h) & \mu^2_{\mathbf{A}}(v', v) \end{pmatrix}.
\end{aligned}$$

Natural transformations of A_∞-functors can now be characterised as "directed homotopies", in the sense explained by the following lemma.

Lemma 4.8. *Let $\mathbf{A}, \mathbf{B}$ be dg-categories. Let $F, G \colon \mathbf{A} \to \mathbf{B}$ be A_∞-functors. There is a bijection between the set of (closed, degree 0) natural transformations $F \to G$ and the set of A_∞-functors $\varphi \colon \mathbf{A} \to \underline{\operatorname{Mor}}\, \mathbf{B}$ such that $S\varphi = F$ and $T\varphi = G$:*

$$\varphi^d = (F^d, G^d, h^d) \leftrightarrow h^d. \tag{4.16}$$

Proof. Let $\varphi \colon \mathbf{A} \to \underline{\operatorname{Mor}}\, \mathbf{B}$ an A_∞-functor as in the hypothesis. In particular, for any string of composable maps $f_1, \ldots, f_d$ with first source X_0 and final target X_d, we have

$$\varphi^d(f_d, \ldots, f_1) = (F^d(f_d, \ldots, f_1), G^d(f_d, \ldots, f_1), h^d(f_d, \ldots, f_1))$$

as a morphism $(F(X_0), G(X_0), h^0_{X_0}) \to (F(X_d), G(X_d), h^0_{X_d})$. Notice that $F^d(\ldots)$ and $G^d(\ldots)$ have degree $|f_1| + \ldots + |f_d| + 1 - d$, that is, $✠_d + 1$, whereas $h^d(\ldots)$ has degree $✠_d$. Now, we unwind the equation (4.8) which defines φ. By Example 4.7, we have

$$\mu^1(\varphi^d) = (\mu^1_{\mathbf{B}}(F^d), \mu^1_{\mathbf{B}}(G^d), -\mu^1_{\mathbf{B}}(h^d) + (-1)^{✠_d+1}\mu^2_{\mathbf{B}}(h^0_{X_d}, F^d) - \mu^2_{\mathbf{B}}(G^d, h^0_{X_0})).$$

Moreover:

$$\begin{aligned}
&\mu^2(\varphi^j(f_d, \ldots, f_{d-j+1}), \varphi^{d-j}(f_{d-j}, \ldots, f_1)) \\
&\quad = \mu^2((F^j, G^j, h^j), (F^{d-j}, G^{d-j}, h^{d-j})) \\
&\quad = (\mu^2_{\mathbf{B}}(F^j, F^{d-j}), \mu^2_{\mathbf{B}}(G^j, G^{d-j}), (-1)^{✠_{d-j}+1}\mu^2_{\mathbf{B}}(h^j, F^{d-j}) - \mu^2_{\mathbf{B}}(G^j, h^{d-j})).
\end{aligned}$$

Now, we find out that the left hand side of (4.8), projected to the third component, is equal to the following:

$$\begin{aligned}
-\mu^1_{\mathbf{B}}(h^d(f_d,\dots,f_1)) - \mu^2_{\mathbf{B}}(G^d(f_d,\dots,f_1),h^0_{X_0}) - (-1)^{\maltese_d}\mu^2_{\mathbf{B}}(h^0_{X_d},F^d(f_d,\dots,f_1))\\
-\sum_{j=1}^{d-1}\mu^2_{\mathbf{B}}(G^j(f_d,\dots,f_{d-j+1}),h^{d-j}(f_{d-j},\dots,f_1))\\
-\sum_{j=1}^{d-1}(-1)^{\maltese_{d-j}}\mu^2_{\mathbf{B}}(h^j(f_d,\dots,f_{d-j+1}),F^{d-j}(f_{d-j},\dots,f_1)).
\end{aligned}$$

We immediately notice that the above term is equal to $-A^d$ in (4.13) if $|h|=0$ there. Moreover, the right hand side of (4.8), projected to the third component, is equal to $-B^d$ in (4.14) if $|h|=0$ there. Now, it is clear that any A_∞-functor $\varphi\colon \mathbf{A}\to \underline{\operatorname{Mor}}\,\mathbf{B}$ such that $S\varphi = F$ and $T\varphi = G$ defines a closed degree 0 natural transformation $h\colon F\to G$ by taking the projection of φ to the third component; conversely, given a closed and degree 0 natural transformation $h\colon F\to G$, setting

$$\varphi^d = (F^d, G^d, h^d)$$

defines an A_∞-functor with the desired properties. Clearly, these mappings are mutually inverse. Moreover, the scrict unitality condition (4.6) for φ is clearly equivalent to the strict unitality condition (4.11) for h. □

If $\mathbf{A}$ and $\mathbf{B}$ are dg-categories, then so is $\operatorname{Fun}_\infty(\mathbf{A},\mathbf{B})$ (this follows immediately from the definition of the higher order compositions). Actually, this is an incarnation of the internal hom $\mathbb{R}\underline{\operatorname{Hom}}(\mathbf{A},\mathbf{B})$:

Proposition 4.9 (([8, Paragraph 4.3], [7, Theorem 0.3]))**.** *Assume that* $\mathbf{A}$ *and* $\mathbf{B}$ *are dg-categories. The dg-category* $\mathbb{R}\underline{\operatorname{Hom}}(\mathbf{A},\mathbf{B})$ *is naturally isomorphic in* Hqe *to the dg-category* $\operatorname{Fun}_\infty(\mathbf{A},\mathbf{B})$ *of strictly unital* A_∞*-functors from* $\mathbf{A}$ *to* $\mathbf{B}$*.*

A sketchy description of the above isomorphism is as follows. Recall from Proposition 2.9 that $H^0(\mathbb{R}\underline{\operatorname{Hom}}(\mathbf{A},\mathbf{B}))$ can be described as the category $\operatorname{qrep}^r(\mathbf{A},\mathbf{B})$ of right quasi-representable $\mathbf{A}$-$\mathbf{B}$-dg-bimodules. Given any dg-category $\mathbf{A}$ there is a dg-category $U(\mathbf{A})$ called the *enveloping dg-category* of $\mathbf{A}$, and a natural quasi-equivalence $U(\mathbf{A})\to\mathbf{A}$ which induces a bijection between A_∞-functors $\mathbf{A}\to\mathbf{B}$ and dg-functors $U(\mathbf{A})\to\mathbf{B}$. Now, it can be proved that $H^0(\operatorname{Fun}_\infty(\mathbf{A},\mathbf{B}))$ is equivalent to $\operatorname{qrep}^r(U(\mathbf{A}),\mathbf{B})$), which in turn is equivalent to $\operatorname{qrep}^r(\mathbf{A},\mathbf{B})$ since $U(\mathbf{A})$ is isomorphic to $\mathbf{A}$ in Hqe. More precise details can be found in [7, Propositions 1.10 and 1.18].

When we identify $H^0(\mathbb{R}\underline{\operatorname{Hom}}(\mathbf{A},\mathbf{B}))$ with $H^0(\operatorname{Fun}_\infty(\mathbf{A},\mathbf{B}))$, the functor

$$\Phi^{\mathbf{A}\to\mathbf{B}}\colon H^0(\operatorname{Fun}_\infty(\mathbf{A},\mathbf{B}))\to \operatorname{Fun}(H^0(\mathbf{A}),H^0(\mathbf{B})) \tag{4.17}$$

is described as follows. Given an A_∞-functor $F\colon\mathbf{A}\to\mathbf{B}$, $F=(F^0,F^1,\dots)$, its image

$$\Phi^{\mathbf{A}\to\mathbf{B}}(F)\colon H^0(\mathbf{A})\to H^0(\mathbf{B})$$

(also denoted by $H^0(F)$) is the functor defined by:

$$\begin{aligned}
A &\mapsto F^0(A), \quad A\in\mathbf{A},\\
[f] &\mapsto [F^1(f)], \quad [f]\in H^0(\mathbf{A})(A,B),
\end{aligned}$$

where $[f]$ denotes the cohomology class of $f\in Z^0(\mathbf{A})(A,B)$. Moreover, given a morphism $[h]_{\mu^1}\colon F\to G$ in $H^0(\operatorname{Fun}_\infty(\mathbf{A},\mathbf{B}))$, its image $\Phi^{\mathbf{A}\to\mathbf{B}}([h]_{\mu^1})$ (also denoted by $H^0(h)$ or $\Phi^{\mathbf{A}\to\mathbf{B}}(h)$ dropping parentheses) is defined by

$$X\mapsto [h^0_X]\in H^0(\mathbf{B})(F^0(X),G^0(X)),\quad X\in\mathbf{A}$$

($[h]_{\mu^1}$ denotes the cohomology class of h with respect to the coboundary operator μ^1 of the complex $\mathrm{Fun}_\infty(\mathbf{A},\mathbf{B})(F,G)$).

Recalling Lemma 4.8, the action of the above functor on morphisms can also be viewed in terms of directed homotopies. Given an A_∞-functor $\varphi\colon \mathbf{A} \to \underline{\mathrm{Mor}}\,\mathbf{B}$ such that $S\varphi = F$ and $T\varphi = G$, we may identify the natural transformation $H^0(\varphi) = \Phi^{\mathbf{A}\to\mathbf{B}}(\varphi)\colon H^0(F) \to H^0(G)$ with the ordinary functor

$$H^0(\mathbf{A}) \to \mathrm{Mor}(H^0(\mathbf{B}))$$

obtained by the following composition:

$$H^0(\mathbf{A}) \xrightarrow{\Phi^{\mathbf{A}\to\underline{\mathrm{Mor}}\,\mathbf{B}}(\varphi)} H^0(\underline{\mathrm{Mor}}\,\mathbf{B}) \xrightarrow{(4.15)} \mathrm{Mor}(H^0(\mathbf{B})),$$

where we remind that $\Phi^{\mathbf{A}\to\underline{\mathrm{Mor}}\,\mathbf{B}}(\varphi)$ is the functor induced by φ at the level of the homotopy categories (beware of the potential confusion arising from the fact that φ can be viewed both as an object of $\mathrm{Fun}_\infty(\mathbf{A},\underline{\mathrm{Mor}}\,\mathbf{B})$ and as a morphism in $\mathrm{Fun}_\infty(\mathbf{A},\mathbf{B})$).

4.3. **Uniqueness of dg-lifts.** The goal of this section is to prove a uniqueness result for dg-lifts using the formalism and techniques of A_∞-functors. We will need the following (simplified) obstruction theory result, which can be proved with a direct computation. The analogue (general) result is proved for A_∞-algebras in [9, Corollaire B.1.5].

Lemma 4.10. *Let $\mathbb{E}$ be a $\mathbf{k}$-linear category, let $\mathbf{B}$ be a dg-category, and let $n \geq 2$ be an integer. Suppose that we have a finite sequence $(F^0, F^1, \ldots, F^{n-1})$, where $F^0\colon \mathrm{Ob}\,\mathbb{E} \to \mathrm{Ob}\,\mathbf{B}, X \mapsto F(X) = F^0(X)$ and*

$$F^d\colon \mathbb{E}(X_{d-1},X_d)\otimes\ldots\otimes\mathbb{E}(X_0,X_1) \to \mathbf{B}(F(X_0),F(X_d))[1-d],$$

is a linear map, for all $d = 1,\ldots,n-1$. Assume that (4.9) is satisfied for all $d = 1,\ldots,n-1$. Then, the subexpression

$$\sum_{j=0}^{n-2}(-1)^{\maltese_j}F^{n-1}(f_n,\ldots f_{j+3},\mu^2_{\mathbb{E}}(f_{j+2},f_{j+1}),f_j,\ldots,f_1)$$
$$-\sum_{j=1}^{n-1}\mu^2_{\mathbf{B}}(F^j(f_n,\ldots,f_{n-j+1}),F^{n-j}(f_{n-j},\ldots,f_1))$$

of (4.9) is a $\mu^1_{\mathbf{B}}$-cocycle, for any chain of composable maps $f_1,\ldots,f_n$.

Another key tool in our argument is the following lemma, which we first prove in the dg-framework, and then reinterpret with the A_∞ notations:

Lemma 4.11. *Let $\mathbf{A}$ be a dg-category, here* not *viewed as an A_∞-category. Let (A,B,f) and (A',B',f') be objects of $\underline{\mathrm{Mor}}\,\mathbf{A}$, and let $n \in \mathbb{Z}$ such that*

$$H^{n-1}(\mathbf{A}(A,B')) = 0.$$

Next, assume we are given a closed degree n morphism $(u,v,h)\colon (A,B,f) \to (A',B',f')$. If there exist $\tilde{u}\colon A \to A'$ and $\tilde{v}\colon B \to B'$ such that $u = d\tilde{u}$ and $v = d\tilde{v}$, then there exists $\tilde{h}\colon A \to B'$ such that

$$(u,v,h) = d(\tilde{u},\tilde{v},\tilde{h}).$$

Proof. By hypothesis we have $d(u,v,h) = 0$, in particular

$$dh + (-1)^n(f'u - vf) = 0.$$

Now, $f'u = d(f'\tilde{u})$ and $vf = d(\tilde{v}f)$, and so

$$d(h + (-1)^n(f'\tilde{u} - \tilde{v}f)) = 0.$$

In other words, $h + (-1)^n(f'\tilde{u} - \tilde{v}f)$ is a $(n-1)$-cocycle. Hence, by hypothesis, it is a $(n-1)$-coboundary:

$$h + (-1)^n(f'\tilde{u} - \tilde{v}f) = d\tilde{h}.$$

Finally, we compute:

$$d\begin{pmatrix} \tilde{u} & 0 \\ \tilde{h} & \tilde{v} \end{pmatrix} = \begin{pmatrix} u & 0 \\ h + (-1)^n(f'\tilde{u} - \tilde{v}f) + (-1)^{n-1}(f'\tilde{u} - \tilde{v}f) & v \end{pmatrix} = \begin{pmatrix} u & 0 \\ h & v \end{pmatrix}.$$

□

Lemma 4.12. *Let* $\mathbf{A}$ *be a dg-category, now viewed as an* A_∞*-category. Let* (A, B, f) *and* (A', B', f') *be objects in* $\mathbf{Q} = \underline{\mathrm{Mor}}\,\mathbf{A}$ *(viewed as an* A_∞*-category), and let* $n \in \mathbb{Z}$ *such that*

$$H^{n-1}(\mathbf{A}(A, B')) = 0.$$

Next, assume that we are given a degree n *morphism* $(u, v, h)\colon (A, B, f) \to (A', B', f')$ *such that* $\mu^1_{\mathbf{Q}}(u, v, h) = 0$*. If there exist* $\tilde{u}\colon A \to A'$ *and* $\tilde{v}\colon B \to B'$ *such that* $u = \mu^1_{\mathbf{A}}(\tilde{u})$ *and* $v = \mu^1_{\mathbf{A}}(\tilde{v})$*, then there exists* $\tilde{h}\colon A \to B'$ *such that*

$$(u, v, h) = \mu^1_{\mathbf{Q}}(\tilde{u}, \tilde{v}, \tilde{h}).$$

Proof. Recall Example 4.7. $u = \mu^1_{\mathbf{A}}(\tilde{u})$ means $(-1)^{n-1}u = d\tilde{u}$, and $(-1)^{n-1}v = d\tilde{v}$. Apply Lemma 4.11 to $(-1)^{n-1}(u, v, h)$:

$$(-1)^{n-1}(u, v, h) = d(\tilde{u}, \tilde{v}, \tilde{h}) = (-1)^{n-1}\mu^1_{\mathbf{Q}}(\tilde{u}, \tilde{v}, \tilde{h}),$$

and the claim follows. □

We prove the following claim, which is actually a lifting result of natural transformations. From now on, we shall set $F(A) = H^0(F)(A)$ if $F\colon \mathbf{A} \to \mathbf{B}$ is a quasi-functor and $A \in \mathbf{A}$.

Proposition 4.13. *Let* $\mathbb{E}$ *be a* $\mathbf{k}$*-linear category, viewed as a dg-category concentrated in degree* 0*, and let* $\mathbf{B}$ *be a dg-category. Let* $F, G\colon \mathbb{E} \to \mathbf{B}$ *be quasi-functors such that*

$$H^j(\mathbf{B}(F(E), G(E'))) = 0, \tag{4.18}$$

for all $j < 0$ *and for all* $E, E' \in \mathbb{E}$*. Let* $\overline{\varphi}\colon H^0(F) \to H^0(G)$ *be a natural transformation. Then, there exists a morphism* $\varphi\colon F \to G$ *in* $H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbb{E}, \mathbf{B}))$ *such that* $H^0(\varphi) = \overline{\varphi}$*.*

Upon identifying $\mathbb{R}\underline{\mathrm{Hom}}(\mathbb{E}, \mathbf{B})$ with $\mathrm{Fun}_\infty(\mathbb{E}, \mathbf{B})$ (Proposition 4.9), Proposition 4.13 is translated to the following:

Proposition 4.14. *Let* $\mathbb{E}$ *be a* $\mathbf{k}$*-linear category, viewed as a dg-category concentrated in degree* 0*, and let* $\mathbf{B}$ *be a dg-category. Let* $F, G\colon \mathbb{E} \to \mathbf{B}$ *be (strictly unital)* A_∞*-functors, satisfying*

$$H^j(\mathbf{B}(F^0(E), G^0(E'))) = 0, \tag{4.19}$$

for all $j < 0$*, for all* $E, E' \in \mathbb{E}$*. Assume* $\overline{\varphi}\colon H^0(F) \to H^0(G)$ *is a natural transformation. Then, there exists a natural transformation of* A_∞*-functors* $\varphi\colon F \to G$ *such that* $H^0(\varphi) = \overline{\varphi}$*.*

Proof. In view of Lemma 4.8, we try to define recursively a A_∞-functor $\varphi\colon \mathbb{E} \to \underline{\mathrm{Mor}}\,\mathbf{B}$ such that $S\varphi = F, T\varphi = G$, and the induced functor

$$\mathbb{E} = H^0(\mathbb{E}) \to \mathrm{Mor}(H^0(\mathbf{B}))$$

is equal to $\overline{\varphi}$. First, we define a map φ^0 on objects: for any $E \in \mathbb{E}$, we set

$$\varphi^0(E) = (F^0(E), G^0(E), \varphi_E),$$

where φ_E is a chosen lift of the given map $\overline{\varphi}_E \colon F^0(E) \to G^0(E)$. Next, we define φ^1 on a given basis (including the identities of all objects) of the space of morphisms. Given an element $f \colon E_0 \to E_1$ of this basis, we set

$$\varphi^1(f) = (F^1(f), G^1(f), h^1(f)),$$

where $h^1(f)$ is a chosen degree -1 morphism such that

$$-\mu^1_{\mathbf{B}}(h^1(f)) = \mu^2_{\mathbf{B}}(G^1(f), \varphi_{E_0}) - \mu^2_{\mathbf{B}}(\varphi_{E_1}, F^1(f)).$$

$h^1(f)$ exists by the hypothesis that $\overline{\varphi} \colon H^0(F) \to H^0(G)$ is a natural transformation. Moreover, we may choose $h^1(1_E) = 0$ for all $E \in \mathbb{E}$. By construction, $\varphi^1(f)$ is a closed degree 0 morphism in $\mathbf{Q} = \underline{\mathrm{Mor}}\, \mathbf{B}$ (see Example 4.7): this means that (4.9) is satisfied for $d = 1$ (and F there replaced by φ). Furthermore, $\varphi^1(1_E) = 1_{\varphi^0(E)}$.

Now, for $d \geq 2$, assume that we have defined a sequence of maps $(\varphi^1, \dots, \varphi^{d-1})$ satisfying (4.9) (F there replaced by φ) and strict unitality, with

$$\varphi^k(f_k, \dots, f_1) = (F^k(f_k, \dots, f_1), G^k(f_k, \dots, f_1), h^k(f_k, \dots, f_1)).$$

Given composable maps $f_i \colon E_{i-1} \to E_i$ in our chosen basis for $i = 1, \dots, d$, by Lemma 4.10 the expression

$$\begin{aligned} C^d = \sum_{n=0}^{d-2} &(-1)^{\maltese_n} \varphi^{d-1}(f_d, \dots f_{n+3}, \mu^2_{\mathbb{E}}(f_{n+2}, f_{n+1}), f_n, \dots, f_1) \\ &- \sum_{j=1}^{d-1} \mu^2_{\mathbf{Q}}(\varphi^j(f_d, \dots, f_{d-j+1}), \varphi^{d-j}(f_{d-j}, \dots, f_1)) \end{aligned} \tag{4.20}$$

is a $\mu^1_{\mathbf{Q}}$-cocycle $(F^0(E_0), G^0(E_0), \varphi_{E_0}) \to (F^0(E_d), G^0(E_d), \varphi_{E_d})$ in $\underline{\mathrm{Mor}}\, \mathbf{B}$, of degree $1-(d-1) = 2-d$. Since F and G are A_∞-functors, we know that the first two components of C^d are the coboundaries of $F^d(f_d, \dots, f_1)$ and $G^d(f_d, \dots, f_1)$:

$$C^d = (\mu^1_{\mathbf{B}}(F^d(f_d, \dots, f_1), \mu^1_{\mathbf{B}}(G^d(f_d, \dots, f_1), \cdots).$$

Then, the condition (4.19) allows us to apply Lemma 4.12 (with $n = 2-d$). We may choose $h^d(f_d, \dots f_1)$ such that

$$C^d = \mu^1_{\mathbf{Q}}(F^d(f_d, \dots, f_1), G^d(f_d, \dots, f_1), h^d(f_d, \dots, f_1)).$$

So, defining

$$\varphi^d(f_d, \dots, f_1) = (F^d(f_d, \dots, f_1), G^d(f_d, \dots, f_1), h^d(f_d, \dots, f_1))$$

we get the correct identity (4.9). Notice that, if one of the f_i is an identity morphism, then expression (4.20) vanishes, so in that case we may choose $h^d(f_d, \dots, f_1) = 0$, and hence $\varphi^d(f_d, \dots, f_1) = 0$, which is the strict unitality condition. Finally, our result follows by induction. □

Finally, we obtain the following theorem, which is the announced uniqueness result for dg-lifts:

Theorem 4.15. *Let $\mathbb{E}$ be a $\mathbf{k}$-linear category, viewed as a dg-category concentrated in degree 0, and let $\mathbf{B}$ be a triangulated dg-category. Let $F, G \colon \mathbb{E} \to \mathbf{B}$ be quasi-functors, such that*

$$H^j(\mathbf{B}(F(E), F(E'))) = 0, \tag{4.21}$$

for all $j < 0$, for all $E, E' \in \mathbb{E}$. Let $\overline{\varphi} \colon H^0(F) \to H^0(G)$ be a natural isomorphism. Then, there exists an isomorphism $\varphi \colon F \to G$ in $H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbb{E}, \mathbf{B}))$ such that $H^0(\varphi) = \overline{\varphi}$.

In particular, set $\mathbf{A} = \mathrm{per}_{\mathrm{dg}}(\mathbb{E})$, and view $\mathbb{E}$ as a full dg-subcategory of $\mathbf{A}$; if $F, G \colon \mathbf{A} \to \mathbf{B}$ are quasi-functors satisfying (4.21), then $H^0(F_{|\mathbb{E}}) \cong H^0(G_{|\mathbb{E}})$ implies $F \cong G$ in $H^0(\mathbb{R}\underline{\mathrm{Hom}}(\mathbf{A}, \mathbf{B}))$.

Proof. Since $H^0(F) \cong H^0(G)$ and $\mathbf{B}$ is triangulated, in particular it is closed under shifts up to homotopy, hence (4.18) holds. Then, the proof is a direct application of Proposition 4.13 and Proposition 3.3. The second part of the statement follows from Lemma 3.2. □

5. Applications

In this section we describe an application of the above technique which gives uniqueness results of Fourier-Mukai kernels. The dg-categories of interest in these applications are enhancements of Verdier quotients of the form $\mathsf{D}(\mathbb{A})/L$, where $\mathbb{A}$ is a $\mathbf{k}$-linear category and L is a full subcategory of $\mathsf{D}(\mathbb{A})$ satisfying suitable hypotheses. More precisely, we will work in the framework of the following result.

Lemma 5.1 (([12, Theorem 2.1], [10, Section 6, first two paragraphs]))**.** *Let $\mathbb{A}$ be a $\mathbf{k}$-linear category, viewed as a dg-category. Let $L \subseteq \mathsf{D}(\mathbb{A})$ be a localising subcategory (namely, strictly full triangulated and closed under direct sums), generated by the objects of $L \cap \mathsf{D}(\mathbb{A})^c$, where $\mathsf{D}(\mathbb{A})^c$ denotes the full subcategory of compact objects of $\mathsf{D}(\mathbb{A})$. Then, the full subcategory L^c of compact objects of L is equal to $L \cap \mathsf{D}(\mathbb{A})^c$, and there is a canonical functor*

$$\iota\colon \mathbb{A} \hookrightarrow \mathsf{D}(\mathbb{A})^c \to \mathsf{D}(\mathbb{A})^c/L^c \hookrightarrow (\mathsf{D}(\mathbb{A})/L)^c, \tag{5.1}$$

where $\mathbb{A} \hookrightarrow \mathsf{D}(\mathbb{A})^c = \operatorname{per}(\mathbb{A})$ is induced by the derived Yoneda embedding of $\mathbb{A}$ and the composition of the last two maps is the restriction of the quotient functor $\mathsf{D}(\mathbb{A}) \to \mathsf{D}(\mathbb{A})/L$. The triangulated category $(\mathsf{D}(\mathbb{A})/L)^c$, together with the natural fully faithful functor $\mathsf{D}(\mathbb{A})^c/L^c \hookrightarrow (\mathsf{D}(\mathbb{A})/L)^c$, is the idempotent completion of $\mathsf{D}(\mathbb{A})^c/L^c$, and it is classically generated by the full subcategory with objects $\iota(\mathbb{A})$.

Moreover, if $\mathbf{D}$ together with the equivalence

$$\epsilon\colon (\mathsf{D}(\mathbb{A})/L)^c \to H^0(\mathbf{D})$$

is an enhancement of $(\mathsf{D}(\mathbb{A})/L)^c$, then $\mathbf{D}$ is quasi-equivalent to $\operatorname{per}_{\mathrm{dg}}(\mathbb{A}')$, where $\mathbb{A}'$ is the full dg-subcategory of $\mathbf{D}$ whose object are given by $\epsilon(\iota(\mathbb{A}))$.

Verdier quotients such as $\mathsf{D}(\mathbb{A})^c/L^c$ are enhanced by the *Drinfeld dg-quotient*. We state its definition and main properties, which we will need in the following.

Definition 5.2 (([6, 1.6.2]))**.** Let $\mathbf{A}$ be a dg-category, and let $\mathbf{B}$ be a full dg-subcategory of $\mathbf{A}$. A *dg-quotient* of $\mathbf{A}$ modulo $\mathbf{B}$ is a dg-category $\mathbf{A}/\mathbf{B}$ together with a quasi-functor $\pi\colon \mathbf{A} \to \mathbf{A}/\mathbf{B}$, such that for any dg-category $\mathbf{C}$ the induced functor

$$\pi^*\colon H^0(\mathbb{R}\underline{\operatorname{Hom}}(\mathbf{A}/\mathbf{B}, \mathbf{C})) \to H^0(\mathbb{R}\underline{\operatorname{Hom}}(\mathbf{A}, \mathbf{C})) \tag{5.2}$$

is fully faithful, and its essential image consists of quasi-functors $F\colon \mathbf{A} \to \mathbf{C}$ such that $H^0(F)$ maps objects of $\mathbf{B}$ to zero objects in $H^0(\mathbf{C})$.

Theorem 5.3 (([6, 1.6.2], [16], [10, Lemma 1.5]))**.** *Let $\mathbf{A}$ be a dg-category, and let $\mathbf{B}$ be a full dg-subcategory of $\mathbf{A}$. A dg-quotient $(\mathbf{A}/\mathbf{B}, \pi)$ exists, and it is uniquely determined up to natural isomorphism in Hqe. Moreover, if $\mathbf{A}$ is pretriangulated and $H^0(\mathbf{B})$ is a triangulated subcategory of $H^0(\mathbf{A})$, then $\mathbf{A}/\mathbf{B}$ is pretriangulated and $(H^0(\mathbf{A}/\mathbf{B}), H^0(\pi))$ is a Verdier quotient of $H^0(\mathbf{A})$ modulo $H^0(\mathbf{B})$:*

$$H^0(\mathbf{A})/H^0(\mathbf{B}) \xrightarrow{\sim} H^0(\mathbf{A}/\mathbf{B}). \tag{5.3}$$

Remark 5.4. Assume the framework of Lemma 5.1. We know that the category $\mathsf{D}(\mathbb{A})^c$ has $\operatorname{per}_{\mathrm{dg}}(\mathbb{A})$ as a dg-enhancement. Moreover, taking $\mathcal{L}^c$ to be the full dg-subcategory of $\operatorname{per}_{\mathrm{dg}}(\mathbb{A})$ whose objects correspond to L^c, we find out that the dg-quotient $\operatorname{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c$ is an enhancement of $\mathsf{D}(\mathbb{A})^c/L^c$. Moreover, since $(\mathsf{D}(\mathbb{A})/L)^c$ can be viewed as the idempotent completion of $\mathsf{D}(\mathbb{A})^c/L^c$, we find out that the dg-category

$$\operatorname{per}_{\mathrm{dg}}(\operatorname{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c)$$

is an enhancement of $(\mathsf{D}(\mathbb{A})/L)^c$. Without loss of generality, we may assume that the functor (5.1) is equal to $H^0(\tilde{\iota})$, where $\tilde{\iota}$ is the quasi-functor

$$\tilde{\iota}\colon \mathbb{A} \hookrightarrow \mathrm{per}_{\mathrm{dg}}(\mathbb{A}) \xrightarrow{\pi} \mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c \hookrightarrow \mathrm{per}_{\mathrm{dg}}(\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c) \tag{5.4}$$

The quasi-functor $\mathrm{per}_{\mathrm{dg}}(\mathbb{A}) \xrightarrow{\pi} \mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c$ is the projection to the dg-quotient, and the fully faithful dg-functors $\mathbb{A} \hookrightarrow \mathrm{per}_{\mathrm{dg}}(\mathbb{A})$ and $\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c \hookrightarrow \mathrm{per}_{\mathrm{dg}}(\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c)$ are induced by the dg-Yoneda embedding, see (2.9). They satisfy the universal properties encoded respectively by (5.2) and (2.10).

Now, [10, Theorem 2.8] tells us that, under the vanishing hypothesis

$$(\mathsf{D}(\mathbb{A})/L)(\iota(A), \iota(A')[j]) = 0, \quad \forall\, j < 0, \quad \forall\, A, A' \in \mathbb{A}, \tag{5.5}$$

the category $(\mathsf{D}(\mathbb{A})/L)^c$ admits a *unique* dg-enhancement ([10, Definition 2.2]). In that case, up to isomorphism in Hqe, we are allowed to identify any such enhancement $\mathbf{D}$ with the dg-category $\mathrm{per}_{\mathrm{dg}}(\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c)$.

Now, the abstract result of the previous section allows us to prove the following:

Theorem 5.5. *Assume the framework of Lemma 5.1, and assume that $(\mathsf{D}(\mathbb{A})/L)^c$ has a unique enhancement. Let $\mathbf{D}$ be such an enhancement, and for simplicity identify $H^0(\mathbf{D}) = (\mathsf{D}(\mathbb{A})/L)^c$. Let $F, G\colon \mathbf{D} \to \mathbf{B}$ be quasi-functors taking values in a triangulated dg-category $\mathbf{B}$, satisfying the vanishing hypothesis:*

$$H^0(\mathbf{B})(F(\iota(A)), F(\iota(A'))[j]) = 0, \quad \forall\, j < 0, \tag{5.6}$$

for all $A, A' \in \mathbb{A}$. Then, if

$$H^0(F) \circ \iota \cong H^0(G) \circ \iota \colon \mathbb{A} \to H^0(\mathbf{B}),$$

we have that $F \cong G$ as quasi-functors.

Proof. Recalling Remark 5.4, we are allowed to identify $\mathbf{D}$ with $\mathrm{per}_{\mathrm{dg}}(\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c)$. By the universal property of $\mathrm{per}_{\mathrm{dg}}(\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c)$, we have that $F \cong G$ if and only if $F_{|\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c} \cong G_{|\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c}$. Then, by the universal property of the dg-quotient, this is equivalent to

$$F_{|\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c} \circ \pi \cong G_{|\mathrm{per}_{\mathrm{dg}}(\mathbb{A})/\mathcal{L}^c} \circ \pi \colon \mathrm{per}_{\mathrm{dg}}(\mathbb{A}) \to \mathbf{B}.$$

Finally, by the universal property of $\mathrm{per}_{\mathrm{dg}}(\mathbb{A})$, this is equivalent to

$$F \circ \tilde{\iota} \cong G \circ \tilde{\iota} \colon \mathbb{A} \to \mathbf{B}.$$

Now, recalling that we have identified $\iota = H^0(\tilde{\iota})$, a direct application of Theorem 4.15 gives the desired result. □

The above result has an interesting application. Let $X \subseteq \mathbb{P}^N$ be a quasi-projective scheme, viewed as open subscheme of a projective scheme $\overline{X}$; for all $n \in \mathbb{Z}$ we shall denote by $\mathcal{O}_X(n)$ (resp. $\mathcal{O}_{\overline{X}}(n)$) the restriction on X (resp. $\overline{X}$) of the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^N$. Following [10, before Corollary 7.8], we introduce a category $\mathbb{A}$ and a subcategory L of $\mathsf{D}(\mathbb{A})$ such that $\mathsf{D}(\mathbb{A})/L$ is equivalent to $\mathfrak{D}(\mathrm{QCoh}(X))$. Namely, take $\mathbb{A}$ as the category with objects given by the integers, and

$$\mathbb{A}(i, j) = H^0(\overline{X}, \mathcal{O}_{\overline{X}}(j - i)), \tag{5.7}$$

with composition induced by that of the graded algebra $\bigoplus_n H^0(\overline{X}, \mathcal{O}_{\overline{X}}(n))$. The subcategory L is taken to be the category of all objects in $\mathsf{D}(\mathbb{A})$ whose cohomologies are "I-torsion modules". Now, by [10, Lemma 7.2, sentence before Corollary 7.8] there is a natural equivalence $\mathsf{D}(\mathbb{A})/L \cong \mathfrak{D}(\mathrm{QCoh}(X))$ such that the composition

$$\mathbb{A} \hookrightarrow \mathsf{D}(\mathbb{A}) \to \mathsf{D}(\mathbb{A})/L \xrightarrow{\sim} \mathfrak{D}(\mathrm{QCoh}(X))$$

maps any integer $j \in \operatorname{Ob}\mathbb{A}$ to the sheaf $\mathcal{O}_X(j)$, which is compact in $\mathfrak{D}(\operatorname{QCoh}(X))$. By [10, Lemma 7.10] the subcategory L satisfies the hypotheses of Lemma 5.1, and in particular the above discussion restricts to compact objects and perfect complexes. Namely, we have an equivalence $(\mathsf{D}(\mathbb{A})/L)^c \cong \operatorname{Perf}(X)$ such that composition with the functor (5.1) gives:

$$\begin{aligned}&\mathbb{A} \xrightarrow{\iota} (\mathsf{D}(\mathbb{A})/L)^c \xrightarrow{\sim} \operatorname{Perf}(X),\\ &j \mapsto \mathcal{O}_X(j).\end{aligned} \tag{5.8}$$

Now, let $\mathfrak{D}_{\mathrm{dg}}(\operatorname{QCoh}(X))$ be an enhancement of $\mathfrak{D}(\operatorname{QCoh}(X))$, and for simplicity identify this category with $H^0(\mathfrak{D}_{\mathrm{dg}}(\operatorname{QCoh}(X)))$. Clearly, the full dg-subcategory $\operatorname{Perf}_{\mathrm{dg}}(X)$ of $\mathfrak{D}_{\mathrm{dg}}(\operatorname{QCoh}(X))$ whose objects are the compact objects in $\mathfrak{D}(\operatorname{QCoh}(X))$ is an enhancement of $\operatorname{Perf}(X)$; also, recall that these enhancements are uniquely determined, by [10, Corollary 7.8, Theorem 7.9]. Upon identifying $(\mathsf{D}(\mathbb{A})/L)^c$ with $\operatorname{Perf}(X)$ via the equivalence discussed above and functor (5.8) with $\iota\colon \mathbb{A} \to (\mathsf{D}(\mathbb{A})/L)^c = \operatorname{Perf}(X)$, as a consequence of Theorem 5.5 we immediately get the following:

Corollary 5.6. *Let X be a quasi-projective scheme, and let $\mathbf{B}$ be a triangulated dg-category. Let $F, G\colon \operatorname{Perf}_{\mathrm{dg}}(X) \to \mathbf{B}$ be quasi-functors which satisfy the vanishing condition*

$$H^0(\mathbf{B})(F(\mathcal{O}_X(n)), F(\mathcal{O}_X(m))[j]) = 0, \quad \forall\, j < 0,$$

for all $n, m \in \mathbb{Z}$. Then, if $H^0(F) \circ \iota \cong H^0(G) \circ \iota$, we have that $F \cong G$ as quasi-functors. In particular, $H^0(F) \cong H^0(G)$ implies $F \cong G$.

Finally, we apply this machinery to the uniqueness problem of Fourier-Mukai kernels, as explained in Section 1, hence obtaining the following uniqueness result:

Theorem 5.7. *Let X and Y be schemes satisfying the hypotheses of Theorem 1.2, with X quasi-projective. Let $\mathcal{E}, \mathcal{E}' \in \mathfrak{D}(\operatorname{QCoh}(X \times Y))$ be such that*

$$F := \Phi_{\mathcal{E}}^{X\to Y} \cong \Phi_{\mathcal{E}'}^{X\to Y}\colon \operatorname{Perf}(X) \to \mathfrak{D}(\operatorname{QCoh}(Y)),$$

and $\operatorname{Hom}(F(\mathcal{O}_X(n)), F(\mathcal{O}_X(m))[j]) = 0$ for all $j < 0$ and all $n, m \in \mathbb{Z}$. Then $\mathcal{E} \cong \mathcal{E}'$.

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Dipartimento di Matematica "F. Casorati", Università di Pavia, Via Ferrata 5, 27100 Pavia (PV), Italy

Email address: francesco.genovese01@ateneopv.it