Lecture 12 (final): "What next?"

1. More general derived categories and dg-enhancements

↳ What is an abelian category $\mathcal{A}$? Essentially, it is a linear ($\mathbb{Z}$-linear, in some cases k-linear) category which behaves like $\mathrm{Mod}(R)$ (R a ring) or $\mathrm{Mod}(\mathcal{O})$ ($\mathcal{O}$ a sheaf of rings), or suitable subcategories thereof such as $\mathrm{QCoh}(\mathcal{O})$ (Quasi-coherent sheaves of $\mathcal{O}$-modules).

Definition: Let $\mathcal{A}$ be a linear category. It is ABELIAN if:

- it has zero objects and finite direct sums.
- it has kernels and cokernels. What are they?

↳ Let $f: A \to B$ in $\mathcal{A}$. The kernel of f is an object $\ker(f) \in \mathcal{A}$ together with a morphism $k: \ker(f) \to A$, such that

$$0 \to \mathcal{A}(-,\ker(f)) \xrightarrow{k_*} \mathcal{A}(-,A) \xrightarrow{f_*} \mathcal{A}(-,B)$$

is exact. Equivalently, $\ker(f) \to A$ satisfies the following universal property:

(i) $\ker(f) \xrightarrow{k} A \xrightarrow{f} B$ is zero;

(ii) For any $K \xrightarrow{q} A \xrightarrow{f} B$ which is zero, $\exists!$ factorization:

$$\begin{tikzcd} K \arrow[r, "q"] \arrow[d, dashed, "\exists!"'] & A \arrow[r, "f"] & B \\ \ker(f) \arrow[ur, "k"'] & & \end{tikzcd}$$

Dually, the cokernel of f is an object $\mathrm{coker}(f) \in \mathcal{A}$ together with a morphism $c: B \to \mathrm{coker}(f)$, such that

$$0 \to \mathcal{A}(\mathrm{coker}(f),-) \xrightarrow{c^*} \mathcal{A}(B,-) \xrightarrow{f^*} \mathcal{A}(A,-)$$

is exact.

[Exercise: write down the explicit universal property].

- The natural morphism

$$\mathrm{coker}(\ker(f)) \to \ker(\mathrm{coker}(f))$$

is an isomorphism, for any $f: A \to B$.

[How do you understand this last property? Think: $\mathrm{coker}(\ker(f)) = A/\ker(f)$

$\ker(\mathrm{coker}(f)) = \mathrm{Im}(f)$

In abelian groups, you have indeed a natural isomorphism $A/\ker(f) \xrightarrow{\sim} \mathrm{Im}(f)$, and in an abelian category this holds axiomatically.]

⇝ Abelian categories behave formally like categories of modules.

- You have injective maps (f s.t. $\mathrm{Ker}(f)=0$), surjective maps ($f: A \to B$ s.t. $\mathrm{Im}(f)=B$), and a morphism is an iso iff it is both injective and surjective.
- You can define complexes of objects and cohomology of such complexes.

⇝ Given your abelian category $\mathcal{A}$, you can define the dg-category of complexes $C_{dg}(\mathcal{A})$ by mimicking what we've done with dg-modules.

↳ With some more work, we can define the derived dg-category $D_{dg}(\mathcal{A})$ and the derived category $D(\mathcal{A})$ (clearly, $D(\mathcal{A}) = H^0(D_{dg}(\mathcal{A}))$). Simplifying the story a little bit, we assume that $\mathrm{Ob}(D_{dg}(\mathcal{A})) = \mathrm{Ob}(C_{dg}(\mathcal{A}))$ (complexes of objects)

We may also set:

$$\left.\begin{aligned} D^+_{(dg)}(\mathcal{A}) &= \{M \in D(\mathcal{A}) : H^k(M) = 0,\ k \ll 0\} \\ D^-_{(dg)}(\mathcal{A}) &= \{M \in D(\mathcal{A}) : H^k(M) = 0,\ k \gg 0\} \\ D^b_{(dg)}(\mathcal{A}) &= D^+(\mathcal{A}) \cap D^-(\mathcal{A}) \end{aligned}\right\} \text{full (dg) subcategories of } D_{(dg)}(\mathcal{A})$$

$D^{(+,-,b)}_{dg}(\mathcal{A})$ is a pretriangulated dg-category ($\equiv$ quasi-equivalent to a strongly pretriangulated dg-category)

⇝ $D^{(+,-,b)}(\mathcal{A})$ is a triangulated category.

↳ Algebraic geometers like derived categories of the form: $D(\underbrace{\mathrm{QCoh}(X)}_{\text{quasi-coherent } \mathcal{O}_X\text{-modules}})$, $D^b(\underbrace{\mathrm{Coh}(X)}_{\text{coherent } \mathcal{O}_X\text{-modules}})$ (X a suitable scheme), and moreover

$\mathrm{Perf}(X)$: perfect complexes of $\mathcal{O}_X$-modules ($\equiv$ locally quasi-isomorphic to bounded complexes of locally free sheaves of finite rank).

↳ We recall that, for a given triangulated category $\mathcal{T}$, we say that $\mathcal{T}$ has a unique dg-enhancement if there is a pretriangulated dg-category $\mathcal{A}$ s.t. $H^0(\mathcal{A}) \cong \mathcal{T}$, and for any other pretriangulated dg-category $\mathcal{B}$ s.t. $H^0(\mathcal{A}) \cong H^0(\mathcal{B})$, we have that $\mathcal{A}$ and $\mathcal{B}$ are quasi-equivalent.

Theorem (Canonaco, Neeman, Stellari 2021): All derived categories $D^{(+,-,b)}(\mathcal{A})$ have unique dg-enhancements.
- $\mathrm{Perf}(X)$ has a unique dg-enhancement if X is quasi-compact, quasi-separated.

To my knowledge, there are still open problems!

↳ Often, categories like $D^b(\mathrm{Coh}(X))$ have 'semiorthogonal decompositions':

$D^b(\mathrm{Coh}(X)) = \langle \mathcal{T}_1, \mathcal{T}_2 \rangle$,

- $\mathcal{T}_1, \mathcal{T}_2 \subseteq D^b(\mathrm{Coh}(X))$ full triangulated subcategories
- $\mathrm{Hom}(\mathcal{T}_2, \mathcal{T}_1) = 0$,
- For $F \in D^b(\mathrm{Coh}(X))$, $\exists$ distinguished triangle: $X_2 \to F \to X_1$; $X_2 \in \mathcal{T}_2$, $X_1 \in \mathcal{T}_1$.

We know that $D^b(\mathrm{Coh}(X))$ has a unique dg-enhancement. Do $\mathcal{T}_1, \mathcal{T}_2$ have unique dg-enhancements?

2. Higher algebra of dg-algebras

↳ If A is a ring, it is useful to study the (abelian) category $\mathrm{Mod}(A)$ of (right) A-modules.

↳ If R is a dg-algebra, it will likely be useful to study its derived category $D(R)$

↳ Interesting case: R (as a complex) is concentrated in nonpositive degrees.

In this case, $D(R)$ has an additional structure, called t-structure

$$\left.\begin{array}{l} D(R)_{\leq n} = \{M \in D(R) : H^k(M) = 0,\ k > n\} \\ D(R)_{\geq n} = \{M \in D(R) : H^k(M) = 0,\ k < n\}. \end{array}\right\} \text{ this pair of subcategories is the t-structure}$$

- $\mathrm{Hom}_{D(R)}(X, Y) = 0$ if $X \in D(R)_{\leq n}$, $Y \in D(R)_{\geq n+1}$
- For any $X \in D(R)$, there is a distinguished triangle: $\tau_{\leq n} X \to X \to \tau_{\geq n+1} X$,
  where $\tau_{\leq n} X \in D(R)_{\leq n}$, $\tau_{\geq n+1} X \in D(R)_{\geq n+1}$.

smart truncations

Important observation: $D(R)^{\heartsuit} \doteq D(R)_{\leq 0} \cap D(R)_{\geq 0} \cong \mathrm{Mod}(H^0(R)) \hookrightarrow D(R)$.

heart of the t-structure

↳ Thanks to the t-structure, we can define INJECTIVE/PROJECTIVE OBJECTS in $D(R)$, also called derived injectives/projectives.

Definition: $P \in D(R)$ is (derived) projective if:

- $P \in D(R)_{\leq 0}$
- $\underbrace{\mathrm{Hom}_{D(R)}(P, Z[1])}_{\doteq \mathrm{Ext}^1_{D(R)}(P,Z)} = 0 \quad \forall Z \in D(R)_{\leq 0}$

Dually, $I \in D(R)$ is (derived) injective if

- $I \in D(R)_{\geq 0}$
- $\underbrace{\mathrm{Hom}_{D(R)}(Z[-1], I)}_{= \mathrm{Ext}^1_{D(R)}(Z,I)} = 0, \quad \forall Z \in D(R)_{\geq 0}$

Some consequences:

- $P \in D(R)$ projective $\Rightarrow$

$$H^0(-) : \mathrm{Hom}_{D(R)}(P, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Mod}(H^0(R))}(H^0(P), H^0(X)) \quad \text{is an isomorphism.}$$

In particular, $H^0(P) \in \mathrm{Mod}(H^0(R))$ is projective.

Dually, $I \in D(R)$ injective $\Rightarrow$

$$H^0(-): \mathrm{Hom}_{D(R)}(X, I) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Mod}(H^0(R))}(H^0(X), H^0(I)) \text{ is an isomorphism}$$

In particular, $H^0(I)$ is injective in $\mathrm{Mod}(H^0(R))$

Example:

R (as right dg-module over itself) is projective in $D(R)$.

Indeed, by the Yoneda lemma: $\mathrm{Hom}_{D(R)}(R, Z[1]) = H^0(Z[1]) = H^1(Z) = 0$ if $Z \in D(R)_{\leq 0}$.

↳ We can do PROJECTIVE/INJECTIVE RESOLUTIONS of objects in $D(R)$.

Let's have a look at projective resolutions:

We resolve objects in $D^-(R) = \{ M \in D(R) : H^k(M) = 0,\ k \gg 0 \}$.

Upon shifting, assume that $M \in D^-(R)$ lies in $D(R)_{\leq 0}$.

- Consider $H^0(M) \in \mathrm{Mod}(H^0(R))$. Find a surjection:

$$H^0(R)^{\oplus I_0} \twoheadrightarrow H^0(M).$$

- Lift it uniquely to a morphism $R^{\oplus I_0} \xrightarrow{d_0} M$.
- Take the distinguished triangle: $C_0 \to R^{\oplus I_0} \xrightarrow{d_0} M$.
- Take a surjection $H^0(R)^{\oplus I_1} \twoheadrightarrow H^0(C_0)$ and lift it uniquely to $R^{\oplus I_1} \to C$
- Consider the commutative diagram, where the rows are distinguished triangles:

$$\begin{tikzcd} C_0 \arrow[r] & R^{\oplus I_0} \arrow[r, "d_0"] & M \\ R^{\oplus I_1} \arrow[u] \arrow[r] & R^{\oplus I_0} \arrow[u, equal] \arrow[r] & X_1 \arrow[u, dashed, "d_1"'] \end{tikzcd}$$

← a first approximation

- Then, iterate: $C_1 \to X_1 \xrightarrow{d_1} M$. Find a surjection $H^0(R)^{\oplus I_2} \to H^{-1}(C_1)$, lift it to $R^{\oplus I_2}[1] \to C_1$

And:

$$\begin{tikzcd} C_1 \arrow[r] & X_1 \arrow[r] & M \\ R^{\oplus I_2}[1] \arrow[u] \arrow[r] & X_1 \arrow[u, equal] \arrow[r] & X_2 \arrow[u, dashed, "d_2"'] \end{tikzcd}$$

Inductively, we have:

$$\begin{tikzcd} R^{\oplus I_0} = X_0 \arrow[r] \arrow[drr] & X_1 \arrow[r] \arrow[dr, "d_1"'] & X_2 \arrow[r] \arrow[d, "d_2"] & \cdots \arrow[r] & X_n \arrow[r] \arrow[dll, "d_n"] & \cdots \\ & & M & & & \end{tikzcd}$$

We can check: $H^{-n}(X_n) \twoheadrightarrow H^{-n}(M)$ is surjective; $H^{-i}(X_n) \xrightarrow{\sim} H^{-i}(M)$ is an isomorphism for $i < n$.

- X_n is an ((n+1)-fold) iterated cone
- One can completely recover M by taking a suitable colimit of $X_0 \to X_1 \to \cdots$. This colimit is a "more general version" of a projective resolution.
- Indeed, we can completely recover $D^-(R)$ (w/ the t-structure) by its (derived) projective objects. (→ 'T-structures and twisted complexes on derived injectives', —, Lowen, Van den Bergh)

↳ work in progress on deformations of triangulated categories.